On the Stabilization of Linear Systems Under Assigned I/O Quantization

Bruno Picasso and Antonio Bicchi

Abstract

This paper is concerned with the stabilization of discrete–time linear systems with quantization of the input and output spaces, i.e., when available values of inputs and outputs are discrete. Unlike most of the existing literature, we assume that how the input and output spaces are quantized is a datum of the problem, rather than a degree of freedom in design. Our focus is hence on the existence and synthesis of symbolic feedback controllers, mapping output words into the input alphabet, to steer a quantized I/O system to within small invariant neighborhoods of the equilibrium starting from large attraction basins. We provide a detailed analysis of the practical stabilizability of systems in terms of the size of hypercubes bounding the initial conditions, the state transient and the steady–state evolution. We also provide an explicit construction of a practically stabilizing controller for the quantized I/O case.

Index Terms

Quantized systems, controlled invariance, practical stability, dynamic output feedback

I. Introduction

Quantization is a peculiar characteristic of many systems, which can be caused by A/D and D/A conversion, binary or digital sensors and actuators, etc. In other cases, it is necessary to introduce quantization of signals in order to reduce the information complexity of some sensors (such as e.g., video cameras) by encoding it in a proper symbolic alphabet. Since the work [4], quantized

This work was supported by European Commission through the IST-2004-004536 (IP) "RUNES" and the IST-2004-511368 (NoE) "HYCON".

B.Picasso is with Scuola Normale Superiore, Pisa, Italy and Centro Interdipartimentale di Ricerca E. Piaggio, Università di Pisa, Italy (e–mail: `picasso.bruno@gmail.com`).

A.Bicchi is with Centro Interdipartimentale di Ricerca E. Piaggio, Università di Pisa, Italy (e–mail: `bicchi@ing.unipi.it`).

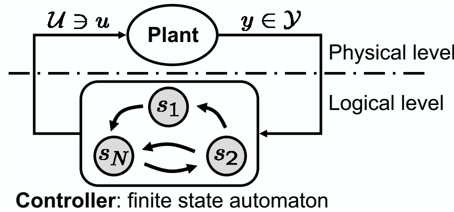

Fig. 1. Graphical illustration of the system considered in this paper.

control systems have been attracting increasing attention of the control community. Most recently, interest on quantization has been stimulated by the growing number of applications involving "networked" control systems, i.e., systems interconnected through communication channels of limited capacity [1], [9], [10], [14], [15].

This paper deals with the control of the dynamical system

$$\begin{cases} x(t+1) = Ax(t) + b\,u(t) \\ y(t) = q\big(x(t)\big) \\ x \in \mathbb{R}^n, \quad u \in \mathcal{U} \subset \mathbb{R}\,, \quad y \in \mathcal{Y}\,, \quad t \in \mathbb{N}\,, \end{cases} \tag{1}$$

where $\mathcal{U}$ is a *given* closed discrete set (i.e., a closed set made of isolated points) and $q : \mathbb{R}^n \to \mathcal{Y}$ is an *assigned* output map taking values in a countable set $\mathcal{Y}$ (finite or infinite). A representation of the control problem for system (1) is depicted in Fig. 1. The overall system has a hybrid structure and is organized into two levels: at the logical level, the controller manipulates output and input strings from discrete alphabets. At the physical level, the plant is modelled by equation (1).

We focus on the stabilization problem for system (1). It has been clarified already in [4] that stability notions most adjusted to quantized systems are "practical stability" properties, involving controlled invariance and attractivity of state space subsets. Accordingly, our study concerns the existence and construction of symbolic feedback controllers capable of steering the system to within minimal invariant neighborhoods of the equilibrium, starting from large attraction basins.

In most of the literature on stabilization, quantization is considered as a degree of freedom in control synthesis: namely, the designer can choose the elements of the control set, as well as the output quantization map. In this framework, the goal usually consists in finding I/O quantization schemes which guarantee the achievement of prefixed stability objectives, fulfilling performance requirements under given communication constraints (see [1], [3], [5]–[10], [14], [15]). Results in this vein have a strong theoretical interest as they allow to identify fundamental limitations in quantized control, but do not provide much insight in the problem with assigned quantization. Indeed, when quantization is given, finding minimal controlled invariant sets is non–trivial. Existing results for this problem either use Lyapunov arguments [3], [4], providing invariant ellipsoids, or use algorithmic approaches [13], resulting in polytopic invariants. Typically, polytopes are less conservative but have a complex structure [2]. A description of invariant sets which is both simple and tight is clearly desirable. Moreover, the nature of the problem varies with the different types of output map q. Results in the literature are mainly concerned with quantization of the state [3], [4] or quantization of the innovation [5], [14]. Some results for quantized outputs are given in [3], [8] but a detailed study on the synthesis of a dynamic symbolic controller for discrete–time systems with arbitrarily assigned quantized outputs has not been addressed so far (to the best of our knowledge).

In this paper a different viewpoint is taken: in fact, the input and output sets $\mathcal{U}$, $\mathcal{Y}$, as well as the output map q, are arbitrarily assigned data of the problem. Control synthesis for stabilization is then subdued to these data. This kind of study is useful because it allows to decide whether a desired control goal can be achieved by using a *given* technology, as e.g., actuators (modelled by $\mathcal{U}$), sensors (modelled by q) and, more in general, communication and computational means.
The main contribution of this paper consists in providing a novel and simple technique to face the practical stabilization problem for quantized SISO systems. Namely, $\mathcal{U} \subset \mathbb{R}$ and the output map is of the type $q = q_o \circ C$, with $C \in \mathbb{R}^{1\times n}$ and $q_o : \mathbb{R} \to \mathcal{Y}$. A peculiarity of our approach consists in studying the problem in terms of invariant hypercubes in the controller form coordinates, which provide a simple yet tight description of minimal invariant sets (cf. [12]). The basic observation is that most of the significant information on the input set $\mathcal{U}$, the output map q and the dynamics of the system, are contained in a pair (ρ, H) of scalar functions providing an effective representation of the resolution (or *dispersion*) of the quantizers. The problem is then reduced to the study of these functions: the invariance of a hypercube of edge length Δ and the possibility of converging to within a smaller hypercube can be tested by means of algebraic relations involving the pair $\big(\rho(\Delta), H(\Delta)\big)$ and the infinity norm of the matrix A.
Although we focus on the stabilization under *assigned* quantization, our technique can be applied also to *design* the quantizers so as to achieve desired stability properties. In this case, our results prove to be conservative in terms of bit–rate with respect to the tight bound proved in [9], [10], [14]. This is however inherently related with the faster convergence our control strategy ensures.

The paper is organized as follows: the problem is formulated in Section II, the main results about practical stability are presented in Section III which is concluded by an example. Due to space limitations, the easy proofs of the most technical results are omitted and can be found in [11].

Notation: $Q_n(\Delta) := \big[-\frac{\Delta}{2};\frac{\Delta}{2}\big]^n$ is the hypercube of edge length Δ. Let $E \subseteq \mathbb{R}^k$: E^{ch} and $\# E$ denote its convex hull and its cardinality, respectively; $\mathrm{diam}(E) := \sup_{x,y\in E} \|x-y\|_2$ is the diameter of E. Given $v \in \mathbb{R}^k$, $E+v := \{x \in \mathbb{R}^k \,|\, x - v \in E\}$. Let x_i be the i^{th} coordinate of x: given $\Omega \subseteq \mathbb{R}^n$, $\mathrm{Pr}_i(\Omega) := \{\omega_i \,|\, \omega \in \Omega\}$ and $\mathrm{diam}_i(\Omega) := \mathrm{diam}\big(\mathrm{Pr}_i(\Omega)\big)$. If $A \in \mathbb{R}^{n\times n}$, $A\,\Omega = \{A\omega \,|\, \omega \in \Omega\}$ while $(Ax)_i$ is the i^{th} coordinate of the vector Ax. x' denotes the transpose of the vector x, $x^+(t)$ stands for $x(t+1)$: the dependance on t will be often omitted.

II. Preliminaries

We deal with the single–input, discrete–time, quantized linear system described in (1). Throughout the paper we suppose that the pair (A,b) is reachable and, without loss of generality, that it is in *controller form*: let $s^n - \alpha_n s^{n-1} - \cdots - \alpha_2 s - \alpha_1$ be the characteristic polynomial of A. Because $\|A\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^n |A_{i,j}| = \max\big\{1\,,\, \sum_{i=1}^n |\alpha_i|\big\}$, if $\sum_{i=1}^n |\alpha_i| \le 1$ then the system is stable, we hence assume $\sum_{i=1}^n |\alpha_i| > 1$ and we let $\alpha := \sum_{i=1}^n |\alpha_i|$.

The output quantizer $q:\ \mathbb{R}^n \to \mathcal{Y}$ is characterized by the induced state space partition: $\mathbb{R}^n = \bigcup_{y\in\mathcal{Y}} \mathcal{C}_y$, where $\mathcal{C}_y := q^{-1}(y)$. We assume that the partition is locally finite, namely, if $\mathcal{B} \subset \mathbb{R}^n$ is bounded, $\#\{y \in \mathcal{Y} \,|\, \mathcal{B} \cap \mathcal{C}_y \neq \emptyset\} < +\infty$. We are mainly interested in the *quantized output* case, where $q = q_o \circ C$ with $C \in \mathbb{R}^{1\times n}$, $q_o : \mathbb{R} \to \mathcal{Y}$ and (A,C) is an observable pair. Two other cases will be considered which are themselves interesting, have stronger properties and are instrumental for the quantized output case: the *quantized state* case, where the state space partition is assumed to be made of bounded sets (at least in a sufficiently large neighborhood of the equilibrium), and a relaxation of system (1), denoted by $(A,b,\mathcal{U})$, in which quantization is on inputs only while full state is available, namely $q\big(x(t)\big) = x(t)$ and $\mathcal{Y} = \mathbb{R}^n$.

With regard to the control set, we assume that $0 \in \mathcal{U}$. Given $\widehat{\mathcal{U}} \subseteq \mathcal{U}$, let $\rho_{\widehat{\mathcal{U}}}$ represent the dispersion (or maximal gap) of $\widehat{\mathcal{U}}$, that is:

$$\rho_{\widehat{\mathcal{U}}} := \begin{cases} \sup\big\{\mathrm{diam}\big(\,]a;b[\,\big) \,\big|\,]a;b[\subseteq \widehat{\mathcal{U}}^{\,\mathrm{ch}} \quad \text{and} \quad]a;b[\cap \widehat{\mathcal{U}} = \emptyset\,\big\} & \text{if } \#\widehat{\mathcal{U}} > 1 \\ +\infty \quad \text{(conventionally)} & \text{otherwise.} \end{cases} \tag{2}$$

Definition 1: Given system (1), let

$$\nu:\ \mathbb{R} \longrightarrow \mathcal{U} \tag{3}$$

be a map which associates to each real number r an element of $\mathcal{U}$ minimizing the Euclidean distance from r. The feedback law $k : \mathbb{R}^n \to \mathcal{U}$ defined by

$$k(x) := \nu\big(-\textstyle\sum_{i=1}^{n} \alpha_i x_i\big) = \nu\big(-(Ax)_n\big)$$

is called *quantized deadbeat controller* (qdb–controller).

As $\mathcal{U}$ is a closed set, the definition of ν is well posed. We shall often implicitly use the following property: if $S \subset \mathbb{R}^k$ is a closed discrete set and $\mathcal{B} \subset \mathbb{R}^k$ is bounded, then $\#(\mathcal{B} \cap S) < +\infty$.

The practical stability properties we will study are strictly related to the notion of invariant set [2]:

Definition 2: The set $\Omega \subseteq \mathbb{R}^n$ is said to be *positively invariant* for a system $x^+ = f(x)$ iff $\forall\, x \in \Omega$, $x^+ \in \Omega$.

Definition 3: The set $\Omega \subseteq \mathbb{R}^n$ is said to be *q–controlled invariant* for system (1) iff $\forall\, y \in q(\Omega)$, $\exists\, u \in \mathcal{U}$ such that $\forall\, x \in q^{-1}(y) \cap \Omega$, $x^+ = Ax + b\,u \in \Omega$.

For system $(A, b, \mathcal{U})$, q–controlled invariance coincides with controlled invariance [2].

In the more general case, we will consider dynamical controllers and we will be interested in the practical stability notion of (X_0, X_1, Ω)–*stability*. More precisely, let the controller be described by the following system defined on some set $\mathcal{W}$:

$$\begin{cases} w(t+1) = \gamma\big(w(t), y(t), t\big) \\ u(t) = k\big(w(t), y(t), t\big)\,, \end{cases} \tag{4}$$

where $\gamma : \mathcal{W} \times \mathcal{Y} \times \mathbb{N} \to \mathcal{W}$ and $k : \mathcal{W} \times \mathcal{Y} \times \mathbb{N} \to \mathcal{U}$. The corresponding closed–loop system is:

$$\begin{cases} x(t+1) = Ax(t) + b\,k\Big(w(t), q\big(x(t)\big), t\Big) \\ w(t+1) = \gamma\,\Big(w(t), q\big(x(t)\big), t\Big)\,. \end{cases} \tag{5}$$

Definition 4: (Cf. [6]) Let Ω, X_0 and X_1 be subsets of $\mathbb{R}^n$ such that Ω and X_0 are neighborhoods of the origin and $X_1 \supseteq X_0$ is bounded. The controller (4) is said to be (X_0, X_1, Ω)–*stabilizing* iff the closed–loop dynamics (5) is so that $\forall\, x(0) \in X_0$ and $\forall\, w(0) \in \mathcal{W}$, $x(t) \in X_1$ $\forall\, t \geq 0$ and $\exists\, \bar{t} \in \mathbb{N}$ such that $\forall\, t \geq \bar{t}$, $x(t) \in \Omega$. In this case, system (5) is said to be (X_0, X_1, Ω)–*stable*.

In the quantized state case it will be sufficient to consider static and time invariant controllers: namely, $\mathcal{W} = \{w\}$ and k is not depending on t, hence the static state feedback $u(\cdot)$ is of the type $k \circ q : \mathbb{R}^n \to \mathcal{U}$ for some $k : \mathcal{Y} \to \mathcal{U}$. For such quantized state case the stronger property of (X_0, X_0, Ω)–stability is obtainable, which will be referred to as (X_0, Ω)–*stability*.

The (X_0,Ω)–stability notion for system $(A,b,\mathcal{U})$ is obtained by replacing q with the identity map in the definition.

Notice that if $k\circ q$ is (X_0,Ω)–stabilizing, then X_0 is positively invariant for system $x^+ = Ax + b(k\circ q)(x)$. Moreover, there exists a controller of the type $k\circ q$ making X_0 positively invariant if and only if X_0 is q–controlled invariant.

III. I/O QUANTIZATION

A. q–controlled invariance

The search for q–controlled invariant sets for system (1) is non–trivial: indeed, most of the theory on invariant sets under constrained control is limited to convex input sets [2]. In this section a family of invariant sets is provided under the general assumptions made on q and $\mathcal{U}$.

Given $\Delta > 0$, let $x \in Q_n(\Delta)$ and $u \in \mathcal{U}$: by the controller form of (A,b),

$$x^+ = \big(x_2,\dots,x_n,\textstyle\sum_i \alpha_i x_i + u\big) \in Q_n(\Delta) \;\Leftrightarrow\; \big|\textstyle\sum_i \alpha_i x_i + u\big| \le \frac{\Delta}{2}\,. \tag{6}$$

Thus, for hypercubes $Q_n(\Delta)$ in controller form coordinates, invariance can be tested considering the n^{th} component only. We take advantage of this property to study the practical stability problem in terms of invariant hypercubes. For a given hypercube $Q_n(\Delta)$, let

$$\mathcal{Y}(\Delta) := q\big(Q_n(\Delta)\big) \subseteq \mathcal{Y}$$

be the set of possible outputs when $x \in Q_n(\Delta)$, and let $\mathcal{C}_y^* := \mathcal{C}_y \cap Q_n(\Delta)$.

The q–controlled invariance of $Q_n(\Delta)$ is tantamount to requiring that $\forall\, y \in \mathcal{Y}(\Delta)$, $\exists\, u \in \mathcal{U}$ such that $A\mathcal{C}_y^* + b\,u \subseteq Q_n(\Delta)$ which, by (6), is equivalent to

$$\forall\, y \in \mathcal{Y}(\Delta)\,,\;\; \exists\, u \in \mathcal{U} \quad \text{such that} \quad \mathrm{Pr}_n(A\mathcal{C}_y^*) + u \subseteq \big[-\tfrac{\Delta}{2}\,;\,\tfrac{\Delta}{2}\big]\,. \tag{7}$$

We seek a characterization of q–controlled invariant hypercubes in terms of algebraic relations between scalar parameters abstracting system (1). To this aim, for $\Delta > 0$ and $y \in \mathcal{Y}(\Delta)$, let $h^*(y) := \mathrm{diam}_n(A\mathcal{C}_y^*)$ and

$$H^*(\Delta) := \max_{y\in\mathcal{Y}(\Delta)} h^*(y)\,.$$

Denote by $c^*_{\sup}(y) := \sup\{\mathrm{Pr}_n(A\mathcal{C}_y^*)\}$, $c^*_{\inf}(y) := \inf\{\mathrm{Pr}_n(A\mathcal{C}_y^*)\}$ and $c^*_{\mathrm{mid}}(y) := \frac{c^*_{\sup}(y)+c^*_{\inf}(y)}{2} = c^*_{\inf}(y) + \frac{h^*(y)}{2}$: these quantities depend on Δ too. Consider also $h(y) := \mathrm{diam}_n(A\mathcal{C}_y)$ and

$$H(\Delta) := \sup_{y\in\mathcal{Y}(\Delta)} h(y)\,.$$

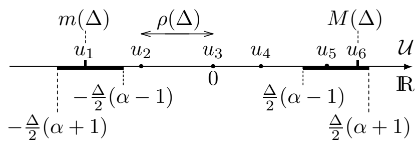

Fig. 2. $\mathcal{U}(\Delta) = \{m(\Delta) = u_1, u_2, u_3, u_4, u_5, u_6 = M(\Delta)\}$: $\rho(\Delta) = u_3 - u_2$, the thicker segments represent the intervals where $m(\Delta)$ and $M(\Delta)$ satisfy inequalities (9).

These parameters are more easily computable than $h^*(y)$ and $H^*(\Delta)$ as they avoid determining the intersection $\mathcal{C}_y^* = \mathcal{C}_y \cap Q_n(\Delta)$, but they lead to a more conservative analysis because $H(\Delta) \geq H^*(\Delta)$. Notice that $H(\Delta)$ and $H^*(\Delta)$ are defined in controller form coordinates, depend on A and are non–decreasing with Δ . These functions are related to the output quantizer resolution. As far as the control set is concerned, we need the following

Lemma 1: If $x \in Q_n(\Delta)$ and u is such that $x^+ \in Q_n(\Delta)$, then $u \in \big[-\frac{\Delta}{2}(\alpha+1)\,;\, \frac{\Delta}{2}(\alpha+1) \big]$.

Proof: See [11] . ■

Hence, the set of the control values that are relevant to ensure the invariance of $Q_n(\Delta)$ is

$$\mathcal{U}(\Delta) := \mathcal{U} \cap \big[-\tfrac{\Delta}{2}(\alpha+1)\,;\, \tfrac{\Delta}{2}(\alpha+1) \big] . \tag{8}$$

Let

$$\begin{cases} m(\Delta) := & \min \mathcal{U}(\Delta) \\ M(\Delta) := & \max \mathcal{U}(\Delta) \end{cases}$$

and, according to (2) , let

$$\rho(\Delta) := \rho_{\mathcal{U}(\Delta)}$$

be the dispersion of $\mathcal{U}(\Delta)$ (see Fig. 2) . The functions $m(\Delta)$, $M(\Delta)$ and $\rho(\Delta)$ depend on the dynamics of the system only through $\|A\|_\infty$. Given $\mathcal{U} \neq \{0\}$, $\exists\, \bar{\Delta} > 0$ such that: for $\Delta < \bar{\Delta}$, $\rho(\Delta) = +\infty$; for $\Delta \geq \bar{\Delta}$, $\rho(\Delta)$ is piecewise constant and non–decreasing with Δ .

Theorem 1: Let $\Delta > 0$, necessary conditions for the q–controlled invariance of $Q_n(\Delta)$ are:

$$m(\Delta) \leq -\tfrac{\Delta}{2}(\alpha-1) \tag{9a}$$

$$M(\Delta) \geq \tfrac{\Delta}{2}(\alpha-1) \tag{9b}$$

$$H^*(\Delta) \leq \Delta \tag{10a}$$

$$\rho(\Delta) \leq \Delta , \tag{10b}$$

if moreover $\rho(\Delta) + H^*(\Delta) \le \Delta$, then $Q_n(\Delta)$ is q–controlled invariant.

To prove Theorem 1 (and other results that will follow), we need the following technical results:

Lemma 2: Let $\Delta > 0$ and ν be the mapping defined in (3). $Q_n(\Delta)$ is q–controlled invariant if and only if one of the following equivalent properties hold:

$\imath$) $\forall\, y \in \mathcal{Y}(\Delta)$, $\exists\, u \in \mathcal{U}$ such that

$$\begin{cases} c^*_{\sup}(y) + u \le \frac{\Delta}{2} \\ c^*_{\inf}(y) + u \ge -\frac{\Delta}{2}\,; \end{cases} \tag{11}$$

$\imath\imath$) $\forall\, y \in \mathcal{Y}(\Delta)$, $\big|c^*_{\text{mid}}(y) + \nu\big(-c^*_{\text{mid}}(y)\big)\big| + \frac{h^*(y)}{2} \le \frac{\Delta}{2}$.

Proof: See [11]. ∎

Lemma 2.$\imath\imath$ points out that the map ν, and therefore the deadbeat controller, has a central role for the invariance problem. Let us show the main properties of ν. To this end, it is helpful to refer to the following notation: $\forall\, \Delta > 0$ such that $\rho(\Delta) < +\infty$, define the partition $\mathbb{R} = \mathcal{S}_{M(\Delta)} \cup \mathcal{N}_\Delta \cup \mathcal{S}_{m(\Delta)}$, where $\mathcal{S}_{M(\Delta)} :=]-\infty\,;\, -M(\Delta) - \frac{\rho(\Delta)}{2}[$, $\mathcal{N}_\Delta := [-M(\Delta) - \frac{\rho(\Delta)}{2}\,;\, -m(\Delta) + \frac{\rho(\Delta)}{2}]$ and $\mathcal{S}_{m(\Delta)} :=]-m(\Delta) + \frac{\rho(\Delta)}{2}\,;\, +\infty[$. Let $\mathcal{S}_\Delta := \mathcal{S}_{M(\Delta)} \cup \mathcal{S}_{m(\Delta)}$. Next Lemma 3 clarifies the relation between the partition $\mathbb{R} = \mathcal{S}_\Delta \cup \mathcal{N}_\Delta$ and the regions where the qdb–controller is or is not saturating.

Lemma 3 (Basic properties of ν): Let $\Delta > 0$:

$\imath$) if inequalities (9) hold, then $\forall\, z \in \text{Pr}_n\big(AQ_n(\Delta)\big)$, $\nu(z) \in \mathcal{U}(\Delta)$;

$\imath\imath$) if $\rho(\Delta) < +\infty$ and $z \in \mathcal{N}_\Delta$, then $|z + \nu(-z)| \le \frac{\rho(\Delta)}{2}$;

$\imath\imath\imath$) assume $\rho(\Delta) < +\infty$ and let z be such that $\nu(-z) \in \mathcal{U}(\Delta)$. If $z \in \mathcal{S}_{M(\Delta)}$, then $\nu(-z) = M(\Delta)$ and $|z + \nu(-z)| = -\big(z + \nu(-z)\big) > \frac{\rho(\Delta)}{2}$; if $z \in \mathcal{S}_{m(\Delta)}$, then $\nu(-z) = m(\Delta)$ and $z + \nu(-z) > \frac{\rho(\Delta)}{2}$.

Proof: See [11]. ∎

Proof of Theorem 1: Since $\text{Pr}_n\big(AQ_n(\Delta)\big) = \big[-\frac{\Delta}{2}\,\alpha\,;\, \frac{\Delta}{2}\,\alpha\big]$ and $\bigcup_{y \in \mathcal{Y}(\Delta)} \mathcal{C}^*_y = Q_n(\Delta)$, then $\exists\, y_{\text{s}} \in \mathcal{Y}(\Delta)$ and $y_{\text{i}} \in \mathcal{Y}(\Delta)$ such that $\forall\, y \in \mathcal{Y}(\Delta)$,

$$\begin{cases} c^*_{\sup}(y) \le c^*_{\sup}(y_{\text{s}}) = \frac{\Delta}{2}\,\alpha \\ c^*_{\inf}(y) \ge c^*_{\inf}(y_{\text{i}}) = -\frac{\Delta}{2}\,\alpha\,. \end{cases} \tag{12}$$

If $Q_n(\Delta)$ is q–controlled invariant, then inequalities (11) hold for $y = y_{\text{s}}$, namely $\exists\, u \in \mathcal{U}$ such that

$$\begin{cases} \frac{\Delta}{2}\,\alpha + u \le \frac{\Delta}{2} \\ c^*_{\inf}(y_{\text{s}}) + u \ge -\frac{\Delta}{2}\,. \end{cases}$$

Since $c^*_{\inf}(y_{\rm s}) \le \frac{\Delta}{2}\,\alpha$, then $\exists\, u \in \mathcal{U}$ such that $-\frac{\Delta}{2}(\alpha+1) \le u \le -\frac{\Delta}{2}(\alpha-1)$, that is $m(\Delta) \le -\frac{\Delta}{2}(\alpha-1)$ (see Fig. 2). The necessity of (9b) can be proved similarly by applying inequalities (11) to $y = y_{\rm i}$.

If $Q_n(\Delta)$ is q–controlled invariant, then $\forall\, y \in \mathcal{Y}(\Delta)$, $h^*(y) \le \Delta$ (as it is clear, for instance, by Lemma 2.$\imath\imath$): the necessity of (10a) hence follows.

To prove the necessity of (10b) we argue by contradiction: if $\rho(\Delta) = +\infty$, then $\mathcal{U}(\Delta) = \{0\}$ but $\|A\|_\infty = \alpha > 1$ implies $AQ_n(\Delta) \not\subseteq Q_n(\Delta)$ which contradicts the invariance of $Q_n(\Delta)$. If instead $\Delta < \rho(\Delta) < +\infty$, then $\exists\, u_1 \in \mathcal{U}(\Delta)$ and $u_2 \in \mathcal{U}(\Delta)$ such that $u_2 - u_1 > \Delta$ and $]u_1\,;\,u_2[\cap \mathcal{U} = \emptyset$. Let $w := \frac{u_1+u_2}{2}$, $w \in \mathrm{Pr}_n\big(AQ_n(\Delta)\big) = \big[-\frac{\Delta}{2}\,\alpha\,;\,\frac{\Delta}{2}\,\alpha\big]$ because $u_2 - u_1 > \Delta$ and, by (8), $u_1 \ge -\frac{\Delta}{2}(\alpha+1)$ and $u_2 \le \frac{\Delta}{2}(\alpha+1)$. Hence, $\exists\, \tilde{x} \in Q_n(\Delta)$ such that $(A\tilde{x})_n = -w$. Let $\tilde{y} := q(\tilde{x}) \in \mathcal{Y}(\Delta)$, since $-w \in \mathrm{Pr}_n(A\mathcal{C}^*_{\tilde{y}})$, then $-w = c^*_{\rm mid}(\tilde{y}) + \theta$ with $|\theta| \le \frac{h^*(\tilde{y})}{2}$. By construction, $|-w+\nu(w)| > \frac{\Delta}{2}$, but $|-w+\nu(w)| \le \big|-w+\nu\big(-c^*_{\rm mid}(\tilde{y})\big)\big| = \big|c^*_{\rm mid}(\tilde{y}) + \theta + \nu\big(-c^*_{\rm mid}(\tilde{y})\big)\big| \le \big|c^*_{\rm mid}(\tilde{y}) + \nu\big(-c^*_{\rm mid}(\tilde{y})\big)\big| + |\theta| \le \big|c^*_{\rm mid}(\tilde{y}) + \nu\big(-c^*_{\rm mid}(\tilde{y})\big)\big| + \frac{h^*(\tilde{y})}{2} \le \frac{\Delta}{2}$, where the first inequality holds because, by definition of ν, $|\nu(r) - w|$ is minimized by $r = w$, while the last one is the invariance condition given in Lemma 2.$\imath\imath$.

Finally, let us show that inequalities (9) and $\rho(\Delta) + H^*(\Delta) \le \Delta$ are sufficient for the q–controlled invariance of $Q_n(\Delta)$. We apply Lemma 2.$\imath\imath$: let $y \in \mathcal{Y}(\Delta)$, if $c^*_{\rm mid}(y) \in \mathcal{N}_\Delta$, then by Lemma 3.$\imath\imath$, $\big|c^*_{\rm mid}(y) + \nu\big(-c^*_{\rm mid}(y)\big)\big| + \frac{h^*(y)}{2} \le \frac{\rho(\Delta)}{2} + \frac{h^*(y)}{2} \le \frac{\rho(\Delta)+H^*(\Delta)}{2} \le \frac{\Delta}{2}$. If instead $c^*_{\rm mid}(y) \in \mathcal{S}_{M(\Delta)}$, since $c^*_{\rm mid}(y) \in \mathrm{Pr}_n\big(AQ_n(\Delta)\big)$, then $\nu\big(-c^*_{\rm mid}(y)\big) \in \mathcal{U}(\Delta)$ (see Lemma 3.$\imath$), and therefore, by Lemma 3.$\imath\imath\imath$, $\big|c^*_{\rm mid}(y) + \nu\big(-c^*_{\rm mid}(y)\big)\big| + \frac{h^*(y)}{2} = \big|c^*_{\rm mid}(y) + M(\Delta)\big| + \frac{h^*(y)}{2} = -\big(c^*_{\rm mid}(y) + M(\Delta)\big) + \frac{h^*(y)}{2} = -c^*_{\inf}(y) - M(\Delta) \le \frac{\Delta}{2}\,\alpha - \frac{\Delta}{2}(\alpha-1) = \frac{\Delta}{2}$, where the inequality follows by inequalities (12) and (9b). The case $c^*_{\rm mid}(y) \in \mathcal{S}_{m(\Delta)}$ is similar. ■

In general, in the quantized output case, $H(\Delta) \equiv +\infty$. In the quantized state case instead, because $H(\Delta) \ge H^*(\Delta)$, we have

Corollary 1: A sufficient condition for the q–controlled invariance of $Q_n(\Delta)$ is that inequalities (9) hold and $\rho(\Delta) + H(\Delta) \le \Delta$. □

Corollary 2: $Q_n(\Delta)$ is controlled invariant for system $(A, b, \mathcal{U})$ if and only if inequalities (9) hold and $\rho(\Delta) \le \Delta$.

Proof: Controlled invariance coincides with q–controlled invariance when q is the identity map. In this case, in condition (7) we can replace "$\forall\, y \in \mathcal{Y}(\Delta)$" with "$\forall\, x \in Q_n(\Delta)$" and "$\mathcal{C}^*_y$" with "$x$". It is then clear that, to obtain the results for controlled invariance, it is sufficient to let

$H^*(\Delta) = 0$. The statement is hence a direct consequence of Theorem 1. ■

B. Closed–loop analysis: quantized state case

In this section we analyze the (X_0, Ω)–stability of the system under the qdb–controller in the quantized state case. We hence assume $H(\Delta) < +\infty$ (at least up to sufficiently large values of Δ). For $y \in \mathcal{Y}(\Delta)$, consider $\mathcal{C}_y$ and let $c_{\text{mid}}(y) := \big(\sup\{\Pr_n(A\mathcal{C}_y)\} + \inf\{\Pr_n(A\mathcal{C}_y)\}\big)/2$ be the middle point of $\Pr_n(A\mathcal{C}_y)$. By suitably redefining q without varying the induced state space partition (hence, without loss of generality), we can suppose that $\mathcal{Y} \subset \mathbb{R}^n$ and $(Ay)_n = c_{\text{mid}}(y)$.

Theorem 2: Let $\Delta > 0$ be such that

$$\begin{cases} m(\Delta) < -\frac{\Delta}{2}(\alpha - 1) & \text{(13a)} \\ M(\Delta) > \frac{\Delta}{2}(\alpha - 1) & \text{(13b)} \\ \rho(\Delta) + H(\Delta) < \Delta\,. & \text{(13c)} \end{cases}$$

The algorithm

$$\begin{cases} \Delta_0 := \Delta\,; \quad \Delta_1 := \rho(\Delta_0) + H(\Delta_0)\,; \quad h := 1\,; \\ \text{while } \big(\rho(\Delta_h) + H(\Delta_h) < \Delta_h\big) \text{ do } \big(\Delta_{h+1} := \rho(\Delta_h) + H(\Delta_h)\,; \quad h := h+1\big) \end{cases} \tag{14}$$

defines a finite sequence $\Delta_0 > \Delta_1 > \cdots > \Delta_f > 0$ and, denoted by $k(x)$ the qdb–controller with saturated inputs $\mathcal{U} = \mathcal{U}(\Delta_0)$, the feedback law $\phi := k \circ q$ is $\big(Q_n(\Delta_0), Q_n(\Delta_f)\big)$–stabilizing.

The proof is based on the following two facts (in particular, Lemma 4 is the key argument):

Lemma 4: Assume that $\Delta > 0$ satisfies inequalities (13) and $(k \circ q)\big(Q_n(\Delta)\big) \subseteq \mathcal{U}(\Delta)$, where $k(x)$ is the qdb–controller. Let $\Delta' := \rho(\Delta) + H(\Delta)$, then $k \circ q$ is $\big(Q_n(\Delta), Q_n(\Delta')\big)$–stabilizing and $(k \circ q)\big(Q_n(\Delta')\big) \subseteq \mathcal{U}(\Delta')$.

Proof: The proof is given after that of Theorem 2. ■

Lemma 5: If $\Delta > 0$ satisfies inequalities (13a-b) and Δ' is such that $\rho(\Delta) \le \Delta' < \Delta$, then Δ' satisfies inequalities (13a-b).

Proof: See [11]. ■

Proof of Theorem 2: The algorithm defines a finite sequence. In fact, if $\rho(\Delta_{h+1}) + H(\Delta_{h+1}) < \rho(\Delta_h) + H(\Delta_h)$, then $\rho(\Delta_{h+1}) < \rho(\Delta_h)$ or $H(\Delta_{h+1}) < H(\Delta_h)$: in the first case $\#\mathcal{U}(\Delta_{h+1}) < \#\mathcal{U}(\Delta_h)$, in the latter $\#\mathcal{Y}(\Delta_{h+1}) < \#\mathcal{Y}(\Delta_h)$, thus $f \le \#\mathcal{U}(\Delta_0) + \#\mathcal{Y}(\Delta_0) < +\infty$.

The $\big(Q_n(\Delta_0), Q_n(\Delta_f)\big)$–stability holds because $\forall\, h = 0, \ldots, f-1$, ϕ is $\big(Q_n(\Delta_h), Q_n(\Delta_{h+1})\big)$–stabilizing. This is ensured by Lemma 4 as its hypotheses are satisfied $\forall\, h = 0, \ldots, f-1$, in fact:

$\rho(\Delta_h)+H(\Delta_h) < \Delta_h$ by construction. Inequalities (13a-b) are satisfied by Δ_h as it follows by recursive application of Lemma 5. Finally, $(k \circ q)\big(Q_n(\Delta_h)\big) \subseteq \mathcal{U}(\Delta_h)$: this holds for $h = 0$ as $\mathcal{U} = \mathcal{U}(\Delta_0)$, while for $h = 1, \ldots, f-1$ it follows by recursive application of Lemma 4. ■

Proof of Lemma 4: We claim that $\forall x \in Q_n(\Delta)$, $|x_n^+| \leq \max\left\{\frac{\rho(\Delta)+H(\Delta)}{2}, \|x\|_\infty - \varphi(\Delta)\right\}$, where $\varphi(\Delta) := \min\left\{M(\Delta) - \frac{\Delta}{2}(\alpha-1), -\frac{\Delta}{2}(\alpha-1) - m(\Delta)\right\}$. This proves Lemma 4, in fact: $\varphi(\Delta) > 0$ by inequalities (13a-b), so the claim implies that $\forall \gamma \in [\Delta' ; \Delta]$, $Q_n(\gamma)$ is positively invariant and, by Lemma 1, $(k \circ q)\big(Q_n(\gamma)\big) \subseteq \mathcal{U}(\gamma)$. Moreover, because $x^+ = (x_2, \ldots, x_n, x_n^+)$, the claim implies that $\|x(n)\|_\infty \leq \max\ \left\{\frac{\Delta'}{2} = \frac{\rho(\Delta)+H(\Delta)}{2}, \|x(0)\|_\infty - \varphi(\Delta)\right\}$: the iteration of this argument yields the $\big(Q_n(\Delta), Q_n(\Delta')\big)$–stability because $\varphi(\Delta) > 0$.

Let us prove the claim. For $x \in Q_n(\Delta)$, with $y = q(x)$, $x_n^+ = (Ax)_n + \nu\big(-(Ay)_n\big)$. The following two properties hold: first, since $(Ay)_n = c_{\text{mid}}(y)$, then $|x_n^+ - y_n^+| \leq \frac{h(y)}{2} \leq \frac{H(\Delta)}{2}$; second, $\nu\big(-(Ax)_n\big) \in \mathcal{U}(\Delta)$ thanks to Lemma 3.$\imath$ which can be applied because inequalities (13a-b) hold. The analysis is then divided into three cases:

I) Suppose that $(Ay)_n \in \mathcal{N}_\Delta$. By Lemma 3.$\imath\imath$, $|y_n^+| \leq \frac{\rho(\Delta)}{2}$, thus $|x_n^+| \leq |x_n^+ - y_n^+| + |y_n^+| \leq \frac{H(\Delta)}{2} + \frac{\rho(\Delta)}{2}$.

II) Suppose that $(Ay)_n \in \mathcal{S}_\Delta$ and x is such that $(Ax)_n \in \mathcal{N}_\Delta$. If $(Ay)_n \in \mathcal{S}_{m(\Delta)}$, $\frac{\rho(\Delta)}{2} - \frac{H(\Delta)}{2} < x_n^+ \leq \frac{\rho(\Delta)}{2}$. In fact: as we assumed $k(y) \in \mathcal{U}(\Delta)$, then $k(y) = m(\Delta)$ by Lemma 3.$\imath\imath\imath$. Thus, $x_n^+ = (Ax)_n + m(\Delta) \leq (Ax)_n + \nu\big(-(Ax)_n\big) \leq \frac{\rho(\Delta)}{2}$ where the first inequality holds because $\nu\big(-(Ax)_n\big) \in \mathcal{U}(\Delta)$ and the latter by Lemma 3.$\imath\imath$. On the other hand, by Lemma 3.$\imath\imath\imath$, $y_n^+ > \frac{\rho(\Delta)}{2}$. Since $|x_n^+ - y_n^+| \leq \frac{H(\Delta)}{2}$ implies $x_n^+ \geq y_n^+ - \frac{H(\Delta)}{2}$, then $x_n^+ > \frac{\rho(\Delta)}{2} - \frac{H(\Delta)}{2}$. The case $(Ay)_n \in \mathcal{S}_{M(\Delta)}$ is similar.

III) Suppose that $(Ay)_n \in \mathcal{S}_\Delta$ and $(Ax)_n \in \mathcal{S}_\Delta$. If $(Ay)_n \in \mathcal{S}_{m(\Delta)}$, we proved in part II that $k(y) = m(\Delta)$ and $x_n^+ > \frac{\rho(\Delta)}{2} - \frac{H(\Delta)}{2}$. Assume that $(Ax)_n \in \mathcal{S}_{M(\Delta)}$, in this case $\frac{\rho(\Delta)}{2} - \frac{H(\Delta)}{2} < x_n^+ < -\frac{\rho(\Delta)}{2}$. In fact: since $\nu\big(-(Ax)_n\big) \in \mathcal{U}(\Delta)$, then $x_n^+ = (Ax)_n + m(\Delta) < (Ax)_n + M(\Delta) = (Ax)_n + \nu\big(-(Ax)_n\big) < -\frac{\rho(\Delta)}{2}$, where both the last equality and the last inequality hold by Lemma 3.$\imath\imath\imath$. If instead $(Ax)_n \in \mathcal{S}_{m(\Delta)}$, then $|x_n^+| \leq \|x\|_\infty - \varphi(\Delta)$. In fact: in this case $k(x) = k(y) = m(\Delta)$ and we can write $m(\Delta) = -\frac{\Delta}{2}(\alpha-1) - \varphi(\Delta) - \theta$, with $\theta \geq 0$. Again by Lemma 3.$\imath\imath\imath$, $x_n^+ = (Ax)_n + m(\Delta) > \frac{\rho(\Delta)}{2} > 0$, hence $|x_n^+| = (Ax)_n + m(\Delta) \leq \sum_i |\alpha_i|\,|x_i| + m(\Delta) \leq \alpha \cdot \|x\|_\infty + m(\Delta) = \alpha \cdot \|x\|_\infty - \frac{\Delta}{2}(\alpha-1) - \varphi(\Delta) - \theta \leq \|x\|_\infty - \varphi(\Delta)$ because $\big(\|x\|_\infty - \frac{\Delta}{2}\big)(\alpha-1) - \theta \leq 0$. The case $(Ay)_n \in \mathcal{S}_{M(\Delta)}$ is similar. ■

Remark 1: The motivation for restricting the qdb–controller to the saturated input set $\mathcal{U}(\Delta_0)$ is that otherwise it might happen that $\phi(x) \notin \mathcal{U}(\Delta_0)$ even if $x \in Q_n(\Delta_0)$ $\big($this could happen because $y = q(x)$ may not belong to $Q_n(\Delta_0)\big)$: in this case $x^+ \notin Q_n(\Delta)$ (see Lemma 1).

Remark 2: Theorem 2 holds also for system $(A, b, \mathcal{U})$: it is sufficient to let $H(\Delta) = 0$ in (13c) and in the algorithm (14). The proof of the theorem can be modified accordingly. Moreover, in this case there is no necessity to saturate the qdb–controller.

Remark 3: Our results can be applied also to design quantization so as to ensure desired stability properties. It is sufficient to select the control set $\mathcal{U}$ and the output map q so that the functions $\rho(\Delta) + H(\Delta)$, $m(\Delta)$ and $M(\Delta)$ satisfy the hypotheses of Corollary 1 (to ensure invariance), or inequalities (13) (to ensure convergence). This can be done by elementary computations. For instance, when $q(x) = x$, a control set ensuring the invariance of $Q_n(\Delta)$ can be constructed according to Corollary 2 : the minimal cardinality of such an $\mathcal{U}$ is approximately $\|A\|_\infty$. Compared with the tight bound proved in [9], [10], [14], which is $\Pi_{|\lambda_i(A)|>1}|\lambda_i(A)|$ (where $\lambda_i(A)$'s are the eigenvalues of A), our result can be conservative: e.g., if A is antistable, $\Pi|\lambda_i(A)| = |\alpha_1| \leq \|A\|_\infty = \sum_{i=1}^n |\alpha_i|$. On the other hand, our controller guarantees good performance in terms of convergence rate. Indeed, our theory can be usefully combined with so–called "zooming" [3] or "nesting" [6] techniques: examples have been provided [6] showing that a nesting version of our controller approaches optimal theoretical bounds relating the number of control values and the speed of convergence $\big($both considered as functions of the contraction Volume(X_0)/Volume$(\Omega)\big)$.

C. *Quantized output case: control synthesis and closed–loop analysis*

The techniques we have introduced allow to deal with quantized outputs $\big($i.e., $q = q_o \circ C$ with $C \in \mathbb{R}^{1\times n}$, $q_o : \mathbb{R} \to \mathcal{Y}$ and (A, C) an observable pair$\big)$. To this aim the controller is endowed with memory and takes the more general form in (4). By storing the past inputs and outputs, it is possible to reconstruct a bounded region within which the current state is confined so that the results presented for the quantized state case can be effectively applied. Nevertheless, such a reconstruction is feasible only after a transient (namely, as soon as the controller has stored enough information), therefore only (X_0, X_1, Ω)–stability can be guaranteed rather than (X_0, Ω)–stability. It is a mild assumption to suppose that $\forall\, y \in \mathcal{Y}$, $q_o^{-1}(y) \subseteq \mathbb{R}$ is a connected set. In this case, there is no loss of generality in assuming that $\mathcal{Y} \subset \mathbb{R}$ and $q_o : \mathbb{R} \to \mathcal{Y}$ is such that $\forall\, y \in \mathcal{Y}$ the closure of $q_o^{-1}(y)$ is either an interval of length λ_y of the type $[\, y - \frac{\lambda_y}{2}\, ;\, y + \frac{\lambda_y}{2}\,]$ or a half–line. Let $\mathcal{Y}_\star := \{y \in \mathcal{Y} \,|\, q_o^{-1}(y) \text{ is an interval of finite length}\}$.

The function $\vec{q}_o : \mathbb{R}^n \to \mathcal{Y}^n$ defined by $\vec{q}_o(z) := \big(q_o(z_1), \ldots, q_o(z_n)\big)$ induces a partition of $\mathbb{R}^n$ such that $\forall\, \vec{y} \in \mathcal{Y}_\star^n$ the closure of $\vec{q}_o^{\,-1}(\vec{y}\,)$ is $\vec{y} + \mathcal{P}_{\vec{y}}$, where $\mathcal{P}_{\vec{y}} = \prod_{i=1}^n [\,-\frac{\lambda_{\vec{y}_i}}{2}\,;\,\frac{\lambda_{\vec{y}_i}}{2}\,]$. Let $S \in \mathbb{R}^{n\times(n-1)}$ be defined by

$$S_{ij} := \begin{cases} 0 & \text{if} \quad i \le j \\ CA^{i-j-1}b & \text{if} \quad i > j\,. \end{cases}$$

Denote by $\vec{u}\,(t)$ and $\vec{y}\,(t)$ the vectors collecting respectively the last $n-1$ inputs and the last n outputs at time t ($t \ge n-1$), that is $\vec{u}\,(t) := \big(u(t-n+1), \ldots, u(t-1)\big)'$, and $\vec{y}\,(t) := \big(y(t-n+1), \ldots, y(t)\big)'$. Let $R := \big[A^{n-2}b\,|\cdots|Ab\,|b\big] \in \mathbb{R}^{n\times(n-1)}$ and $\mathcal{O} \in \mathbb{R}^{n\times n}$ be the observability matrix (i.e., the matrix whose i^{th} row is CA^{i-1}): $\mathcal{O}$ is invertible by hypothesis. By standard theory on observability it holds that $\vec{y}\,(t) = \vec{q}_o\big(\mathcal{O}\,x(t-n+1) + S\vec{u}\,(t)\big)$, hence $x(t-n+1) \in \mathcal{O}^{-1}\Big(\vec{q}_o^{\,-1}\big(\vec{y}\,(t)\big) - S\vec{u}\,(t)\Big)$ and $x(t) \in A^{n-1}\mathcal{O}^{-1}\Big(q_o^{-1}\big(\vec{y}\,(t)\big)\Big) - A^{n-1}\mathcal{O}^{-1}S\vec{u}\,(t) + R\vec{u}\,(t)$. If moreover $\vec{y}\,(t) \in \mathcal{Y}_\star^n$, then the current state belongs to the following bounded set:

$$x(t) \in \mathcal{C}_{x(t)} := A^{n-1}\mathcal{O}^{-1}\big(\mathcal{P}_{\vec{y}(t)}\big) + A^{n-1}\mathcal{O}^{-1}\vec{y}\,(t) + (R - A^{n-1}\mathcal{O}^{-1}S)\,\vec{u}\,(t)\,. \tag{15}$$

Consider the map $\psi : \mathcal{Y}^n \times \mathcal{U}^{n-1} \to \mathbb{R}^n$ defined by $\psi(\,\vec{y}, \vec{u}\,) := A^{n-1}\mathcal{O}^{-1}\vec{y} + (R - A^{n-1}\mathcal{O}^{-1}S)\vec{u}$: if $\vec{y}\,(t) \in \mathcal{Y}_\star^n$, then $\psi\big(\,\vec{y}\,(t)\,, \vec{u}\,(t)\big)$ locates the centroid of the parallelogram $\mathcal{C}_{x(t)}$.

The controller is then defined by selecting the control action as if the current state was $\psi\big(\,\vec{y}\,(t)\,, \vec{u}\,(t)\big)$. Naturally, such controller needs to be initialized for $t \le n-2$, we hence define the *dynamic qdb–controller* as follows: denote by $k(x)$ the standard qdb–controller and let[1]

$$u(t) := \begin{cases} 0 & \text{if} \quad t \le n-2 \\ (k \circ \psi)\big(\vec{y}\,(t), \vec{u}\,(t)\big) & \text{if} \quad t \ge n-1\,. \end{cases} \tag{16}$$

Let us analyze the closed–loop system induced by the dynamic qdb–controller. For $t \le n-1$, by the controller form of A it holds that $\forall\, x(0) \in Q_n(\Delta)$ and $\forall\, t \le n-1$, $x(t) \in Q_n(\Delta\|A^{n-1}\|_\infty)$. For $t \ge n$, in order to use the results presented for the quantized state case, we only need to introduce a measure $\widetilde{H}(\Delta)$ bounding $\text{diam}_n(A\,\mathcal{C}_{x(t)})$: $\forall\,\Delta > 0$ consider $\mathcal{Y}(\Delta) = \big(q_o \circ C\big)\big(Q_n(\Delta)\big)$. If $\mathcal{Y}(\Delta) \subseteq \mathcal{Y}_\star$, define $\Lambda_\Delta := \max_{y\in\mathcal{Y}(\Delta)} \lambda_y$ and $\widetilde{H}(\Delta) := \text{diam}_n\Big(A^n\mathcal{O}^{-1}\big(Q_n(\Lambda_\Delta)\big)\Big)$, else $\widetilde{H}(\Delta) := +\infty$. Let $\Delta > 0$ be such that $\mathcal{Y}(\Delta) \subseteq \mathcal{Y}_\star$ and suppose that $\vec{y}\,(t) \in \mathcal{Y}(\Delta)^n$: since $\mathcal{C}_{x(t)}$ is a translation of the set $A^{n-1}\mathcal{O}^{-1}\big(\mathcal{P}_{\vec{y}(t)}\big)$ $\big($see equation (15)$\big)$, and $\mathcal{P}_{\vec{y}(t)} \subseteq Q_n(\Lambda_\Delta)$, then

$$\text{diam}_n(A\mathcal{C}_{x(t)}) = \text{diam}_n\big(A^n\mathcal{O}^{-1}(\mathcal{P}_{\vec{y}(t)})\big) \le \text{diam}_n\Big(A^n\mathcal{O}^{-1}\big(Q_n(\Lambda_\Delta)\big)\Big) = \widetilde{H}(\Delta)\,.$$

[1]This controller can be modelled as in (4) with $\mathcal{W} := \mathcal{Y}^n \times \mathcal{U}^{n-1}$ and suitably defined mappings γ and k, see [11].

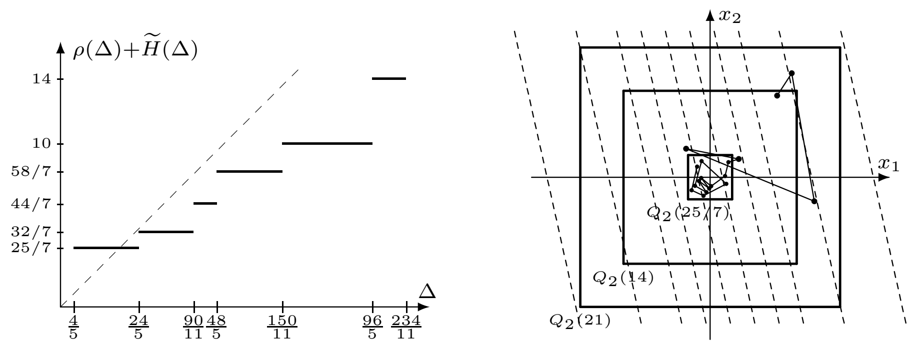

Fig. 3. Example 1. *Left*: graph of $\rho(\Delta)+\widetilde{H}(\Delta)$. *Right*: a trajectory generated by the dynamic qdb–controller for $\Delta_0 = 14$ $(\Delta_1 = 21)$ and $x(0) = (5.42\ 6.60)$. Broken lines identify the state space partition induced by q.

This discussion yields the following

Theorem 3 (Dynamic quantized output feedback): Let $\Delta_1 > 0$ satisfy inequalities (13a-b) and

$$\rho(\Delta_1) + \widetilde{H}(\Delta_1) < \Delta_1 \,.$$

Consider the finite sequence $\Delta_1 > \Delta_2 > \cdots > \Delta_f > 0$ defined by the algorithm (14) $\big($with $\widetilde{H}(\Delta_h)$ in place of $H(\Delta_h)\big)$ and let $\Delta_0 := \frac{\Delta_1}{\|A^{n-1}\|_\infty}$. Then the dynamic qdb–controller (16) with saturated inputs $\mathcal{U} = \mathcal{U}(\Delta_1)$ is $\big(Q_n(\Delta_0), Q_n(\Delta_1), Q_n(\Delta_f)\big)$–stabilizing.

Proof: The proof is similar to that of Theorem 2. The suitable technical adjustments can be found in [11]. ■

Example 1: Consider the unstable system

$$\begin{cases} x^+ = \begin{pmatrix} 0 & 1 \\ 5/4 & 1/4 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y = q_o\Big(\big(\ 3/2 \quad 1/3\ \big)\, x\Big)\,, \end{cases}$$

where $u \in \mathcal{U} = \{0\,,\pm1\,,\pm2\,,\pm3\,,\pm4\,,\pm6\,,\pm8\,,\pm12\,,\pm16\,,\pm24\}$ and the extremes of the intervals forming the output space partition induced by q_o are $\big\{\pm\frac{3}{2}\,,\pm\frac{9}{2}\,,\pm\frac{15}{2}\,,\pm\frac{25}{2}\,,\pm\frac{39}{2}\big\}$.

According to the theory, let $\mathcal{Y} = \mathcal{Y}_\star \cup \{\pm y_{\rm s}\} = \{0\,,\pm3\,,\pm6\,,\pm10\,,\pm16\,,\pm y_{\rm s}\}$ $\big($where $\mathcal{Y}_\star$ collects the middle points of the output quantization intervals and q_o takes the values $\pm y_{\rm s}$ for $|Cx| > \frac{39}{2}\big)$. For $y \in \mathcal{Y}_\star$, the values of λ_y are: $\lambda_0 = \lambda_{\pm3} = \lambda_{\pm6} = 3$, $\lambda_{\pm10} = 5$ and $\lambda_{\pm16} = 7$. Direct computations show that: $\alpha = \|A\|_\infty = \frac{3}{2}$, $\mathcal{U}(\Delta) = \mathcal{U} \cap \big[-\frac{5}{4}\Delta\,;\,\frac{5}{4}\Delta\big]$, thus $\rho(\Delta)$ is found accordingly, and $\widetilde{H}(\Delta) = \frac{6}{7}\Lambda_\Delta$. It ensues that $\rho(\Delta)+\widetilde{H}(\Delta) < \Delta \Leftrightarrow \Delta \in \big]\frac{25}{7}\,;\,\frac{234}{11}\big] := \mathcal{I}$. Since $M\big(\frac{234}{11}\big) = 24 > \frac{234}{11}\cdot\frac{\alpha-1}{2} \simeq 5.32$, by Lemma 5, inequalities (13a-b) are satisfied $\forall\,\Delta \in \mathcal{I}$.

Therefore, Theorem 3 guarantees that $\forall\,\Delta\in\mathcal{I}$, the dynamic qdb–controller with saturated inputs $\mathcal{U}=\mathcal{U}(\Delta)$ is $\big(Q_2(\frac{\Delta}{\alpha}),Q_2(\Delta),Q_2(\Delta_f)\big)$–stabilizing, with $\Delta_f=\frac{25}{7}$ (see Fig. 3).

If quantization can be designed and the goal is to converge into a smaller hypercube $Q_2(\xi)$, while preserving the size of the attraction basin, we can add more input and output values or to redistribute them so that they are less dispersed near 0. As the expressions for $\mathcal{U}(\Delta)$ and $\widetilde{H}(\Delta)$ are available, the problem consists in tuning $\rho(\Delta)$ and Λ_Δ so that $\forall\Delta\in\,\big]\xi\,;\,\frac{234}{11}\big]$, $\rho(\Delta)+\frac{6}{7}\Lambda_\Delta<\Delta$ and inequalities (13a-b) are satisfied for $\Delta=\frac{234}{11}$: this is a simple arithmetic issue.

IV. Conclusion

We have introduced a novel technique for the stabilizability analysis of quantized SISO linear systems. The results hold under very general hypotheses and are of direct applicability. Straightforward extensions to more general control problems are possible, including e.g., the presence of bounded noise terms. Interesting questions are open for future investigations, especially in the framework of sampled continuous–time systems under communication constraints.

Acknowledgements: We are grateful to F. Gouaisbaut, L. Palopoli, F. Fagnani and S. Zampieri for many helpful discussions on the subject of this paper.

References

[1] J. Baillieul, "Feedback Designs in Information–Based Control," In *Stochastic Theory and Control*, B.Pasik–Duncan, Ed. Berlin, Germany: Springer–Verlag, pp. 35–57, 2002.

[2] F. Blanchini, "Set Invariance in Control," *Automatica*, vol. 35, pp. 1747–1767, 1999.

[3] R. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Autom. Control*, vol. 45, no. 7, pp. 1279–1289, Jul. 2000.

[4] D.F. Delchamps, "Stabilizing a Linear System with Quantized State Feedback," *IEEE Trans. Autom. Control*, vol. 35, no. 8, pp. 916–924, Aug. 1990.

[5] N. Elia and S. Mitter, "Stabilization of Linear Systems With Limited Information," *IEEE Trans. Autom. Control*, vol. 46, no. 9, pp. 1384–1400, Sep. 2001.

[6] F. Fagnani and S. Zampieri, "Quantized stabilization of linear systems: complexity versus performance," *IEEE Trans. Autom. Control*, special issue on "Networked Control Systems," vol. 49, no. 9, pp. 1534–1548, Sep. 2004.

[7] M. Fu and L. Xie, "The Sector Bound Approach to Quantized Feedback Control," *IEEE Trans. Autom. Control*, vol. 50, no. 11, pp. 1698–1711, Nov. 2005.

[8] D. Liberzon, "Hybrid feedback stabilization of systems with quantized signals," *Automatica*, vol. 39, pp. 1543–1554, 2003.

[9] Q. Ling and M.D. Lemmon, "Stability of Quantized Control Systems Under Dynamic Bit Assignment," *IEEE Trans. Autom. Control*, vol, 50, no. 5, pp. 734–740, May 2005.

[10] G.N. Nair and R.J. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM Journal on Contr. Optim.* vol. 43, no. 2, pp. 413–436, Feb. 2004.

[11] B. Picasso, "Stabilization of quantized linear systems: analysis, synthesis, performance and complexity," Ph.D. dissertation, Scuola Normale Superiore, Pisa, IT, 2007.

[12] B. Picasso and A. Bicchi, "Hypercubes are minimal controlled invariants for discrete time linear systems with quantized scalar input," *Journal on Nonlinear Analysis: Theory, Methods, and Applications*, special issue on "Hybrid Systems and Applications" (to appear), 2007.

[13] D.E. Quevedo, G.C. Goodwin, J.A. De Doná, "Finite constraint set receding horizon control," *Int. J. of Robust and Nonlinear Control*, vol. 14, no. 4, pp. 355–377, Apr. 2004.

[14] S.C. Tatikonda and S. Mitter, "Control Under Communication Constraints," *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1056–1068, Jul. 2004.

[15] W. Wong and R. Brockett, "Systems with finite communication bandwidth constraints - part II: Stabilization with limited information feedback," *IEEE Trans. Autom. Control*, vol. 44, no. 5, pp. 1049–1053, May 1999.