Reachability Analysis for a Class of Quantized Control Systems

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Abstract

In this paper we study control systems whose input sets are quantized. We specifically focus on problems relating to the structure of the reachable set of such systems, which may turn out to be either dense or discrete. We report on some recent results on the reachable set of linear quantized systems, and study in detail a particular but interesting class of nonlinear systems, forming the discrete counterpart of driftless nonholonomic continuous systems. For such systems, we provide a complete characterization of the reachable set, and, in the case the set is discrete, a computable method to describe its lattice structure.

1 Introduction

In this paper we consider systems of the type

$$x^+ = g(x,u),\ x \in \mathbb{R}^n,\ u \in U \subset \mathbb{R}^m \tag{1}$$

where the input set, U, is quantized, i.e. finite or numerable but nowhere dense in $\mathbb{R}^m$. Quantized control systems arise in a number of applications because of many physical phenomena or technological constraints. In the control literature, quantization of inputs has been considered mainly as due to D/A conversion, and mostly regarded as a disturbance to be rejected ([2, 11, 5]). More recently, some attention has been focused on quantized control systems as specific models of hierarchically organized systems with interaction between continuous dynamics and logic ([13, 6]). More motivations for studying systems with finite input sets come from robotics applications, such as that of manipulating polyhedral parts by rolling ([9]) or steering nonlinear systems by concatenating strings of basic input words ([12, 8]).

The focus of our paper is on the study of particular phenomena that may appear in quantized control systems, which have no counterpart in classical systems theory, and that deeply influence the qualitative properties and performance of the control system. These concern the structure of the set of points that are reachable by system (1), and particularly its density.

While some understanding has been reached recently for the structure of the reachable set for quantized linear systems ([4]) or for specific nonlinear problems arising in robotic manipulation [3], the general nonlinear case remains largely unexplored. In this paper, we address a particular instance of (1), namely driftless nonlinear systems, arising as the discrete counterpart of the chained–form systems systems. Our aim is to report on conditions under which the reachable set for these systems is dense in $\mathbb{R}^n$, or otherwise when it is discrete. In the latter case, the reachable set possesses a lattice structure, whose description by a finitely computable algorithm is instrumental to devising steering methods for the system based on standard integer programming techniques.

2 First definitions and examples

Consider a system defined by a quintuple $(M, \mathcal{T}, U, \Omega, \mathcal{A})$, with M denotes the configuration set, $\mathcal{T}$ an ordered time set, U a set of acceptable input symbols (possibly depending on the configuration), Ω a set of acceptable input words, and $\mathcal{A}$ is the state–transition map $\mathcal{A} : \mathcal{T} \times \Omega \times M \to M$. Denote $\mathcal{A}_{t,\omega}(x) = \mathcal{A}(t,\omega,x)$, with composition by concatenation $\mathcal{A}(t_1,\omega_2,x_1) \circ \mathcal{A}(t_0,\omega_1,x_0) = \mathcal{A}(\mathcal{A}(\mathbf{t}_0,\omega_1,x_0),\omega_2,x_1)$. Explicit dependence of $\mathcal{A}$ on t will be omitted when unnecessary.

In particular, we will focus here on $\mathcal{T} = \mathbb{N}$, as most interesting phenomena relating with quantization appear as linked to discrete time. A system with both M and U discrete sets essentially represents a sequential machine or an automaton, while for M and U continuous sets, a discrete–time, nonlinear control system is obtained. We are interested in studying reachability problems that arise when M has the cardinality of a continuum, but U is discrete (i.e., finite or countable, but nowhere dense), i.e. when inputs

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are *quantized.*

To fix some ideas, consider a discrete time quantized control system in the form

$$x^+ = g(x,u), \tag{2}$$

where $x \in M$, and $u \in U \subset \mathbb{R}^m$, U be quantized, and Ω is the set comprised of all strings of symbols in U. In the following we shall denote by $g_u : M \to M$ the one–step state–transition map $\mathcal{A}_u(\cdot)$ for $u \in U$.

We also denote as R_x the reachable set from x, i.e. the set of configurations x_f for which there exists $\omega = u_1 \cdots u_N \in \Omega$ that steers the system from x to $x_f = g_{u_1} \cdots g_{u_n}(x)$.

For differentiable systems, the notion of *reachability from* x is conventionally understood as $R_x = M$. For discrete–time systems with quantized inputs, however, Ω is a subset of all possible finite sequences ω of symbols in the quantized set U, hence R_x is a countable set and, in the general case that the configuration set has the cardinality of a continuum, it will not make sense checking whether R_x equals M.

Notice that the possibility that the reachable set of a quantized control system is countable, separates such systems from differentiable systems; on the other hand, the possibility of having a dense reachable set distinguishes quantized control systems from classical finite–state machines. We want also to point out that sampled systems with D/A conversions and usage of computers naturally lead to system of type (1) with U finite subset or $\mathbb{Q}^n$. It is then clear that it may be important to describe the structure, and measure the coarseness, of countable reachable sets. To address these concerns, we introduce the further assumption that M is a metric space, and will refer to discreteness or density in the state space of the reachable set.

Let us introduce the relation $\sim$ over the elements of M by setting $x \sim y$, $x, y \in M$, if $y \in R_x$. We want to focus on a special class of systems that we call invertible systems.

Definition 1 The system (2) is said to be invertible if for every $x \in M$ and $u \in U$ there exists a finite sequence of controls $u_i \in U$, $i = 1, \ldots, n$, such that $g_{u_1} \cdots g_{u_n}(g(x,u)) = x$.

The following proposition is obvious:

Proposition 1 *The relation $\sim$ is an equivalence relation if and only if the system is invertible.*

If the system is invertible, we can partition the state space into a family of reachable sets. This is equivalent to take the quotient $M/\sim$ with respect to the equivalence relation $\sim$. We call the set $\widetilde{M} = M/\sim$ the reachability set of the system (2) and we endow $\widetilde{M}$ with the quotient topology, that is the largest topology such that $\pi : M \to \widetilde{M}$, the canonical projection, is continuous.

Example 1. Consider the system

$$x^+ = x + u$$

where $x \in \mathbb{R}$ and $u \in U$, U finite subset of $\mathbb{R}$. If $U = \{0, 1/2, -1\}$ then the system is invertible. The reachable set from the origin R_0 is the subgroup of $\mathbb{R}$ generated by $1/2$ and the reachability set $\widetilde{M}$ is homeomorphic to S^1. If $U = \{\sqrt{2}, -1\}$ then the system is not invertible. For example $\sqrt{2} \in R_0$, but, since $\sqrt{2}$ is irrational, $0 \notin R_{\sqrt{2}}$.

Example2. Consider the system

$$x^+ = g(x,u)$$

where $x \in \mathbb{R}$, $U = \{\pm 1/2, \pm 2\}$ and $g(x,u) = u \cdot x$. The system is invertible, $R_0 = \{0\}$ and for every $x \neq 0$ $R_x = \{\pm 2^i x : i \in \mathbb{Z}\}$. The reachability set $\widetilde{M}$ is homeomorphic to the set $S^1 \cup \{\alpha\}$, where on S^1 there is the usual topology while the only neighborhood of α is the whole space.

Notice that in example 2, the reachable set R_x for $x \neq 0$ has only one accumulation point, namely 0. If we assume that M is a metric space and the maps g_u are isometries then we have a dichotomy illustrated by next proposition:

Proposition 2 *Consider an invertible system (2). Let (M,d) be a metric space and assume that $x \to g(x,u)$ is an isometry for every $u \in U$ then each reachable set R_x is formed either by accumulation points or by isolated points.*

Proof. Assume that the set R_x admits an accumulation point $\bar{x} \in R_x$. Let $x_k \in R_x$ be such that $x_k \to \bar{x}$ and the set $\{x_k : k \in \mathbb{Z}\}$ is infinite. Since the system is invertible, for every k there exists $\tilde{u}_k = (u_k^1, \ldots, u_k^{n_k})$ such that $u_k^i \in U$ and $g_{u_k^1} \cdots g_{u_k^{n_k}}(x_k) = x$. Define $y_k = \lim_m g_{u_k^1} \cdots g_{u_k^{n_k}}(x_m)$. For every k and m we have:

$$\begin{aligned} & d(g_{u_k^1} \cdots g_{u_k^{n_k}}(x_m), x) = \\ & \quad d(g_{u_k^1} \cdots g_{u_k^{n_k}}(x_m), g_{u_k^1} \cdots g_{u_k^{n_k}}(x_k)) = \\ & \quad d(x_m, x_k). \end{aligned}$$

Passing to the limit in m, we have $d(y_k, x) = d(\bar{x}, x_k)$. Clearly the sequence y_k converge to x and contains infinitely many distinct points, so x is an accumulation point for R_x. Now it easily follows that all points of R_x are accumulation points for R_x. $\square$

The system:

$$x^+ = x + u \tag{3}$$

with $x \in R^n$ is an interesting special case. It is clear that for every $x_0 \in R^n$ the reachable set R_{x_0} from x_0 is equal to $x_0 + R_0$ where R_0 is the reachable set from the origin. The hypothesis of the above Proposition are satisfied. Notice that if $n = 1$ and U is symmetric then the set R_0 is either everywhere dense or nowhere dense in $\mathbb{R}$ (since it is a subgroup of $\mathbb{R}$), hence presenting a stronger dichotomy of the one illustrated by the above Proposition. For $n > 1$ we may have directions along which the reachable set R_0 is dense and directions along which is discrete. This is precisely the case of $n = 2$ and $U = \{(\pm 1, 0), (\pm\sqrt{2}, 0), (0, \pm 1)\}$. Notice that if we define $\pi_v : \mathbb{R}^n \to \mathbb{R}$ to be the orthogonal projection on the direction of the vector v, then $\pi_v(R_0)$ is dense in $\mathbb{R}$ for every v not parallel to $(0, 1)$ (and this corresponds to the fact that the projection of the reachable set is precisely the reachable set of the projection of the system). On the other side, $R_0 \cap \{\lambda v : \lambda \in \mathbb{R}\}$ is discrete for every v not parallel to $(1, 0)$.

3 Analysis problems

In this section, we provide some results on the question concerning some simple examples of driftless systems of the type

$$x^+ = x + u \tag{4}$$

where $x \in \mathbb{R}$ and u takes values in a finite set $U \subset \mathbb{R}$. Given two real numbers $r_1, r_2 \in \mathbb{R}$ we write $r_1 \sim r_2$ to indicate that r_1, r_2 have rational ratio, that is $\frac{r_1}{r_2} \in \mathbb{Q}$. It is easy to check that $\sim$ is an equivalence relation.

The following was proven in ([1])

Theorem 1 *Let R_0 be a reachable set for the system (4) from the origin. Then R_0 is dense if and only if there exist $u, v \in U$ such that $u \not\sim v$ and $u \cdot v < 0$. Moreover, if R_0 is not dense then is nowhere dense.*

Since the reachable set from a point x_0 is exactly $x_0 + R_0$ we have a dichotomy similar to that of Section 2, even if in this case (due to the possible lack of symmetry of U) R_0 may fail to be a subgroup of $\mathbb{R}$.

Let us consider the system (4) but now with $x \in \mathbb{R}^n$, that is

$$x^+ = x + u \tag{5}$$

with $x \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}^n$, U a quantized set. From the above analysis, we get for the set R_0 of configurations reachable from the origin for system (5) the following

Theorem 2 *A necessary condition for the reachable set R_0 to be dense is that U contains $n+1$ controls of which n are linearly independent. If $u_1, \ldots, u_n \in U$ are linearly independent and there exists n irrational negative numbers $\alpha_1, \ldots, \alpha_n$ such that $v_i = \alpha_i u_i \in U$ for every $i = 1, \ldots, n$ then R_0 is dense in $\mathbb{R}^n$. If there exists $m \leq n$ vectors v_i such that $\forall u \in U$, there exists m integers $a_1, \ldots, a_m$ such that $u = a_i v_i$, then R_0 is discrete in $\mathbb{R}^n$.*

4 Quantized Chained–Form Systems

We are interested in studying the structure of the reachability set for nonlinear system that exhibit nonholonomic behaviors. To do so, we consider the discrete–time analog of a much studied class of continuous–time nonholonomic systems that are written in chained form

$$\begin{array}{rcl} \dot{x}_1 & = & u_1 \\ \dot{x}_2 & = & u_2 \\ \dot{x}_3 & = & x_2 u_1 \\ \vdots & = & \vdots \\ \dot{x}_n & = & x_{n-1} u_1 \end{array} . \tag{6}$$

The chained form was introduced in [10] because it allows a rather simple steering method, using sinusoids at integrally related frequencies. A different technique for steering continuous nonholonomic systems that are in strictly triangular form[1] has been proposed in [8]. The idea there was to purposefully introduce quantization of the input space, by defining a set of fixed input functions on compact time sets, which resulted in a finite steering algorithm.

Consider now the discrete system

$$\begin{array}{rcl} x_1^+ & = & x_1 + u_1 \\ x_2^+ & = & x_2 + u_2 \\ x_3^+ & = & x_3 + x_2 u_1 + u_1 u_2 \frac{1}{2} \\ x_4^+ & = & x_4 + x_3 u_1 + x_2 u_1^2/2 + u_1^2 u_2 \frac{1}{6} \\ \vdots & = & \vdots \\ x_n^+ & = & \sum_{i=0}^{n-2} x_{n-i} \frac{u_1^i}{i!} + u_1^{n-2} u_2 \frac{1}{(n-1)!} \end{array} \tag{7}$$

which can be regarded as system (6) under unit sampling. Notice that this system is invertible (as opposed e.g. to the forward Euler approximation of (6)). Indeed, for any state–independent, symmetric set of input symbols U, the set of input words

[1] A system is in ST form if $\dot{x}_i = g(x_{i+1}, \cdots, x_n)u$. ST systems include, but are not limited to, nilpotent systems [7], and are hence much more general than chained form systems.

$\Omega = \{\text{strings of symbols in } U\}$ with the relation $\omega\omega^{-1} = \emptyset$, is a group with inverse $(w_1 w_2 \cdots w_m)^{-1} = -w_m \cdots -w_b - w_a$, $\pm w_i \in U, \forall i$. Ω acts on the configuration space through the state–transition map such that $\mathcal{A}(\omega^{-1}, \mathcal{A}(\omega, x)) = x$.

We are interested in studying the reachability set of system (7), and in providing a steering method for the system. Our program is to show first that the reachability analysis in the whole state space $\mathbb{R}^n$ can be decoupled in the reachability analysis in the base space $\mathbb{R}^2$ and in the fiber space corresponding to a given reachable base point, $(\bar{x}_1, \bar{x}_2)$. Reachability in the base space will then be studied by results reported in the previous section, and the rest of the paper will be devoted to the study of reachability in the fiber space.

Few simple calculations show that $\forall \omega \in \Omega$, $\omega = w_1 w_2 \cdots w_N$, it holds

$$\mathcal{A}(\omega, x) = x + A(\omega, x) + \Delta(\omega).$$

By denoting as w_i^j the j–th component of w_i and introducing the shorthand notation $\sigma = \sigma(\omega) = \sum_{i=1}^N w_i^1$, $\tau = \tau(\omega) = \sum_{i=1}^N w_i^2$, $\sigma_i = \sigma_i(\omega) = \sum_{j>i}^N w_j^1$, one can compute, for $j \geq 3$, $A_j(\omega, x) = \sum_{i=2}^{j-1} \frac{1}{(j-i)!} x_i \sigma^{j-i}$. Furthermore, one has, for $j \geq 3$, $\Delta_j(\omega) =$

$$\frac{\sum_{i=1}^N w_i^2}{(j-1)!} \left((w_i^1)^{j-2} + \sigma_i \left(\sum_{k=0}^{j-3} \gamma_{jk} (w_i^1)^k \sigma_i^{j-3-k} \right) \right)$$

where $\forall j \geq 3$, γ_{jk} is given by

$$\gamma_{jk} = \begin{cases} j-1 & \text{for } k = 0, j-3 \\ \gamma_{(j-1)(k-1)} + \gamma_{(j-1)k} & \text{for } k = 1, \ldots, j-4. \end{cases}$$

Consider the subgroup $\tilde{\Omega} \in \Omega$ of control words that take the base variables back to their initial configuration. These are sequences of inputs such that the sum of the first and second components are zero, i.e. $\sigma = \tau = 0$, hence $A(\omega, x) = 0$ and the first two components of $\Delta(\omega)$ are zero.

The action of this subgroup on the fiber is additive: namely, $\mathcal{A}(\tilde{\omega}_1, \mathcal{A}(\tilde{\omega}_2, x)) = \mathcal{A}(\tilde{\omega}_2, x) + \mathcal{A}(\tilde{\omega}_1, x)$, $\forall \tilde{\omega}_1, \tilde{\omega}_2 \in \tilde{\Omega}$. Notice that this represents a significant departure from the behavior of the continuous model (6), where the action of the generic cyclic control is additive only on the first fiber variable, x_3, and more restricted subgroups should be searched that have the additive action property on the rest of the fiber. By the additivity of the action of $\mathcal{A}$ for all $\tilde{\omega} \in \tilde{\Omega}$, one has that $\mathcal{A}(\tilde{\omega}, \cdot)$ is an isometry (w.r.t. the Euclidean norm) on the fiber. Hence, without loss of generality we may study the reachable points along the fiber over $\bar{x}_1 = 0, \bar{x}_2 = 0$. Along any other fiber the reachable points will have the same structure which can be, by Propositions 2 and by the additivity of the action of $\mathcal{A}$, either everywhere or nowhere dense.

System (7) can therefore be decomposed, to the purposes of reachability analysis, in two different discrete systems of the form (5). The first subsystem is simply $y^+ = y + u$ with $y = (x_1, x_2) \in \mathbb{R}^2$ and $u \in U \subset \mathbb{R}^2$. The second subsystem is given by $z^+ = z + v$ with $z = (x_3, x_4, \ldots, x_n) \in \mathbb{R}^{n-2}$ and $v \in \widetilde{U} \subset \mathbb{R}^{n-2}$ where $\widetilde{U} = \{\Delta(\omega),\ \omega \in \tilde{\Omega}\}$ (here and in what follows, we abuse notation and use $\Delta(\omega)$ to denote the $n-2$–dimensional projection of Δ on the fiber space).

Clearly $\widetilde{U}$ is itself symmetric. Indeed if $\omega \in \tilde{\Omega}$ then also $\omega^{-1} \in \tilde{\Omega}$ and $\Delta(\omega^{-1}) = -\Delta(\omega)$. Observe that Theorem 2 can be used in order to estimate the reachable set for $y \in \mathbb{R}^2$. On the other hand, $\widetilde{U}$ is not finite, nor is it known whether it has accumulation points, and hence conditions of Theorem 2 cannot be checked directly.

In what follows, we will consider systems with U finite and symmetric, of cardinality $2c+1$. Namely, let $U = \{0, w^1, \ldots, w^c, \bar{w}^1, \ldots, \bar{w}^c\}$ where by $\bar{w}^j$ we denote $(w^j)^{-1}$. Assume also that $\forall i = 1, \ldots, c$, w^i is a rational number, so that we know from Theorem 2 that the reachable set in the base space is discrete. For such systems, we will show that there exists a finite set B of generators for $\widetilde{U}$, so that the reachable set of $z+ = z + v$ for $v \in \widetilde{U}$ and that for $v \in B$ coincide. Conditions of Theorem 2 are computable for a finite set, and it will be shown that the reachable set in the fiber space is actually discrete as well. Explicit computation of the generators in B would finally lead to a complete description of the lattice structure of the reachable set.

As a first step, a set of generators for $\tilde{\Omega}$ is characterized. Define a map counting the number of appearances of different (signed) symbols in a string of Ω as follows. Let $\hat{\Sigma} : \Omega \to \mathbb{N}^{2c}$ with $\hat{\Sigma}(w_1 \ldots w_N) = (\hat{\beta}_1, \ldots, \hat{\beta}_c, \hat{\beta}_{c+1}, \ldots, \hat{\beta}_{2c})^T$ where $\hat{\beta}_i = \sum_{j=1}^N \delta_{ij}$ and

$$\delta_{ij} = \begin{cases} \begin{cases} 1 & if\ w_j = w^i \\ 0 & otherwise \end{cases} & i = 1, \ldots, c \\ \begin{cases} 1 & if\ w_j = \bar{w}^{(i-c)} \\ 0 & otherwise \end{cases} & i = c+1, \ldots, 2c \end{cases}$$

Further, let $\Sigma : \Omega \to \mathbb{Z}^c$ with $\Sigma(w_1 \ldots w_N) = (\beta_1, \ldots, \beta_c)^T$ where $\beta_i = \hat{\beta}_i - \hat{\beta}_{c+i}$. Σ counts the number of appearances of different unsigned symbols in a string, taking their signs into account.

Remark: For the map Σ the following properties hold

a) if $\omega_1, \omega_2 \in \Omega$ then $\Sigma(\omega_1\omega_2) = \Sigma(\omega_1) + \Sigma(\omega_2)$;

b) for all $\omega \in \Omega$, $\Sigma(\omega^{-1}) = -\Sigma(\omega)$;

c) if $\omega_1 = w_1 \ldots, w_N$ and ω_2 is obtained by permutation of symbols of ω_1, then $\Sigma(\omega_1) = \Sigma(\omega_2)$.

If ω_1 and ω_2 are as in c) the we denote $\omega_2 \equiv \omega_1$. Furthermore,

d) by a),b) and c), if $\omega_1 \equiv \omega_2$ then $\Sigma(\omega_1\omega_2^{-1}) = 0$

Let $U = [w^1 \cdots w^c]$ be the matrix whose columns are the positive elements of U. Let also N_U denote the $c \times (c-2)$–matrix with integer coefficients such that $UN_U = 0$ and, $\forall j = 1, \ldots, c-2$, $G.C.D.\{(N_U)_{ij}, i = 1, \ldots, c\} = 1$.

Proposition 3 $\tilde{\Omega}$ *can be characterized as:*

$$\tilde{\Omega} = \{\omega \in \Omega, s.t. \Sigma(\omega) = (N_U\alpha),\ \alpha \in (\mathbb{N} \cup \{0\})^{c-2}\}.$$

Proof. Let ω be such that $\Sigma(\omega) = (N_U\alpha)$ for some $\alpha \in (\mathbb{N} \cup \{0\})^{c-2}$. Then, collecting together symbols from U,

$$\pi_{\mathbb{R}^2}\mathcal{A}(\omega, x) = \pi_{\mathbb{R}^2}\mathcal{A}(\underbrace{\hat{w}^1 \ldots \hat{w}^1}_{|\beta_1| times} \cdots \underbrace{\hat{w}^c \ldots \hat{w}^c}_{|\beta_c| times}, x)$$

where $\pi_{\mathbb{R}^2} : \mathbb{R}^n \to \mathbb{R}^2$ is the canonical projection of $\mathbb{R}^n$ onto $\mathbb{R}^2$, $(\beta_1, \ldots, \beta_c) = \Sigma(\omega)$ and

$$\hat{w}^i = \begin{cases} w^i & if\ \beta_i > 0 \\ \bar{w}^i & if\ \beta_i < 0 \end{cases}$$

Recalling that $\Sigma(\omega) = (N_U\alpha)$ then $\pi_{\mathbb{R}^2}A(\omega, x) = \pi_{\mathbb{R}^2}(x) + U\Sigma(\omega) = \pi_{\mathbb{R}^2}(x) + UN_U\alpha = \pi_{\mathbb{R}^2}(\mathbf{x})$. Then $\omega \in \tilde{\Omega}$.

Vice versa let $\omega \in \tilde{\Omega}$. Suppose for absurd that $U\Sigma(\omega) \neq 0$. Then by permuting the symbols of ω one has that

$$\omega \equiv \underbrace{\hat{w}^1 \ldots \hat{w}^1}_{|\beta_1| times} \cdots \underbrace{\hat{w}^c \ldots \hat{w}^c}_{|\beta_c| times} = U\beta = U\Sigma(\omega) \neq 0$$

Then $\pi_{\mathbb{R}^2}\mathcal{A}(\omega, x) = \pi_{\mathbb{R}^2}(x) + U\Sigma(\omega) \neq I_{\mathbb{R}^2}(x)$, which is a contradiction. $\square$

Consider now

$$\begin{aligned} \mathcal{L} \;=\; & \{\omega \in \Omega\ s.t. \forall i, (\hat{\beta}_i \neq 0) \Longrightarrow (\hat{\beta}_{i+c} = 0) \\ & s.t.\ \Sigma(\tilde{\omega}) = \pm(N_U)_j, \text{the j–th column of } N_U, \\ & \text{for some } \ j = 1, \ldots, c-m\}. \end{aligned}$$

This subset is comprised of those words of given Σ weight whose $\hat{\Sigma}$ weight is minimal (such words contain no substrings of type $\omega_1\omega_2\omega_1^{-1}$).

Proposition 4 *The set*

$$C = \{\omega\tilde{\omega}\omega^{-1}; \omega \in \Omega, \tilde{\omega} \in \mathcal{L}\}$$

generates $\tilde{\Omega}$. *Hence, the reachable set in the fiber is given by*

$$\Delta_C = \{\Delta(\omega),\ \omega \in C\} \subset \tilde{U}.$$

The proof is given in the appendix. By some computation, it can be observed that $\forall \omega = (w_1 \cdots w_N) \in \Omega, \tilde{\omega} \in \tilde{\Omega}$ one can write $\Delta(\omega\tilde{\omega}\omega^{-1}) = G(\omega)\Delta(\tilde{\omega})$ with

$$G(\omega) = \exp(-J_0\sigma)$$

where J_0 is a $(n-2)$ lower Jordan block with zero eigenvalues. and $\sigma = \sigma(\omega) = \sum_{i=1}^N w_i^1$. Hence, for the generating set it holds $\Delta_C = \{G(\omega)\Delta(\tilde{\omega}), \forall \omega \in \Omega, \tilde{\omega} \in \mathcal{L}\}$. Observe that Δ_C is not yet a finite basis (because Ω is an infinite free group). However a finite basis for Δ_C is provided by a deeper analysis as follows.

Since by hypothesis $w_i \in \mathbb{Q}^2$, let the first component $w_i^1 = \frac{p_i}{q_i}$, with p_i, q_i coprime integers, and let d_i, p, q be integer numbers with p, q coprime and $\frac{p_i}{q_i} = d_i\frac{p}{q}$ $\forall i = 1, \ldots, c$. Then, for some $\alpha_i \in \mathbb{Z}$, one can write $\sigma(\omega) = \sum_{i=1}^N w_i^1 = \sum_{i=1}^c \alpha_i w_i^1 = \frac{p}{q}\sum_{i=1}^c \alpha_i d_i$. Define $k(\omega) \in \mathbb{Z}$ as $k(\omega) = \sum_{i=1}^c \alpha_i d_i$, such that $\sigma(\omega) = \frac{p}{q}k(\omega)$. Observe that $k(\omega) = -k(\omega^{-1})$.

Proposition 5 *Let* $B = \{G(\hat{\omega}_0)\Delta(\tilde{\omega}), \cdots, G(\hat{\omega}_{n-3})\Delta(\tilde{\omega}), \hat{\omega}_i \in \Omega\ s.t.\ k(\hat{\omega}_i) = i$ *and* $\tilde{\omega} \in \mathcal{L}\}$. *Then* B *is a finite set and generates* Δ_C *by integer linear combinations.*

Proof. Fix $\tilde{\omega}$. To prove the proposition it is sufficient to show that for $\omega \in \Omega$ with $k(\omega) > n$ or $k(\omega) < 0$, a positive linear integer combination of $G(\hat{\omega}_0), \cdots, G(\hat{\omega}_{n-3})$ exists such that $\sum_{i=0}^{n-3} \beta_i G(\hat{\omega}_i)\Delta(\tilde{\omega}_i) = G(\omega)\Delta(\tilde{\omega})$. Notice that this is equivalent to showing that a linear combination over the integers exists such that

$$\sum_{i=0}^{n-3} a_i G(\hat{\omega}_i) = G(\omega). \tag{8}$$

since one can take $\beta_i = a_i$, $\tilde{\omega}_i = \tilde{\omega}$ if $a_i \geq 0$, else $\beta_i = -a_i$ and $\tilde{\omega}_i = \bar{\tilde{\omega}}$.

Observe that $G(\hat{\omega}_i)$ is in the form

$$G(\hat{\omega}_i) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots \\ \frac{p}{q}i & 1 & 0 & 0 & \cdots & \cdots \\ \frac{1}{2!}\frac{p^2}{q^2}i^2 & \frac{p}{q}i & 1 & 0 & \cdots & \cdots \\ \frac{1}{3!}\frac{p^3}{q^3}i^3 & \frac{1}{2!}\frac{p^2}{q^2}i^2 & \frac{p}{q}i & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

The fact that such Toeplitz matrices are completely specified by their first column implies that finding the solution of (8) is reduced to solving for the first column, i.e., if $k(\omega) = \nu$, solving the system of $n-2$ equations

$$\sum_{i=0}^{n-3} a_i i^k = \nu^k, \quad k = 0, \ldots, n-3 \tag{9}$$

in a_i, $i = 0, \ldots, n-3$. The unique solution of (9) is in integer. Indeed (9) can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \mu_0 & \mu_1 & \cdots & \mu_{n-3} \\ \mu_0^2 & \mu_1^2 & \cdots & \mu_{n-3}^2 \\ \vdots & \vdots & & \vdots \\ \mu_0^{n-3} & \mu_1^{n-3} & \cdots & \mu_{n-3}^{n-3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-3} \end{bmatrix} = \begin{bmatrix} 1 \\ \nu \\ \nu^2 \\ \nu^3 \\ \cdots \end{bmatrix} \tag{10}$$

where $\mu_i = i$. Observe that the Vandermonde determinant of the matrix in (10) is $\prod_{0 \leq i < j \leq n-3} (\mu_j - \mu_i)$. By the Cramer rule, solutions are given by

$$a_k = \frac{\prod_{0 \leq i < k} (\nu - \mu_i) \prod_{k < j \leq n-3} (\mu_j - \nu) \prod_{\substack{0 \leq i < j \leq n-3 \\ i,j \neq k}} (\mu_j - \mu_i)}{\prod_{0 \leq i < j \leq n-3} (\mu_j - \mu_i)}$$
$$= \frac{\prod_{0 \leq i < k} (\nu - i) \prod_{k < j \leq n-3} (j - \nu)}{\prod_{0 \leq i < k} (k - i) \prod_{k < j \leq n-3} (j - k)}$$

i.e., up to sign, by binomial coefficients, which are integers. □

We have thus obtained a finite set B of generators for $(\widetilde{U})$ and, from application of Theorem 2 to $y+ = y + v$ with $v \in B$, we get that the reachable set is discrete. Furthermore, the reachable set has a lattice structure that can be easily described in terms of integer combinations of generators, which are computed once the initial control set U is given.

Such computation would proceed by first computing $G(\hat{\omega}_i)$, $i = 0, \ldots, n-3$, hence computing the set $\{\Delta(\tilde{\omega});\ \tilde{\omega} \in \mathcal{L}\}$, and combining results. The number of elements $\tilde{\omega}$ in $\mathcal{L}$ is finite, but possibly very large. Indeed, $\forall \tilde{\omega}_a \in \mathcal{L}$, also all $\tilde{\omega}$ obtained by permutation of symbols in $\tilde{\omega}_a$ are in $\mathcal{L}$. The following proposition drastically reduces the number of necessary computations to characterize the reachable set.

Proposition 6 *Let $\omega_1 = w_1 \cdots w_m \cdots w_l \cdots w_N \in \mathcal{L}$ and $\omega_2 = w_1 \cdots w_l \cdots w_m \cdots w_N \in \mathcal{L}$ be obtained by exchanging symbols in the m–th and l–th position. Then $\Delta(\omega_2) = \Delta(\omega_1) + \delta(m, l)$ with*

$$\delta(m, l) = \sum_{i=1}^{l-m} \delta(l-i, l) + \sum_{j=1}^{l-m-1} \delta(m, l-j).$$

The j–th element of $\delta(m, m+1)$ evaluates to
$\delta_j(m, m+1) = D(m) \left(p(j-1, m) + s \sum_{k=2}^{j-2} \frac{s^{j-2-k}}{(j-1-k)!} p(k, m) \right)$,
where $D(m) = det([w_m\ w_{m+1}])$, $s = \sum_{i \geq m+2}^{N} w_i^1$, and
$p(j, m) = \begin{cases} 1 & for j = 2 \\ \frac{1}{j!} \sum_{k=0}^{j-3} \gamma(j,k) (w_m^1)^k (w_{m+1}^1)^{j-3-k} & for\ j \geq 3 \end{cases}$

Proof. The proof is based on simple but lengthy calculations for $\delta(m, m+1)$, and by induction for successive transposition of adjacent symbols in the input string. □

Once a finite set of generators is found, the problem of steering the system to a given reachable configuration can be easily solved by standard techniques of integer linear programming: namely, a change of coordinates is found in which the generators are in Hermite normal form; the steering problem is trivially solved in these coordinates; and finally actual inputs are found by the generalized inverse Euclid algorithm. Other elementary tools of number theory can be profitably used, such as Minkowsky's convex body theorem to establish worst–case errors to reach generic points in $\mathbb{R}^n$.

5 Conclusions

In this paper, we have considered reachability problems in quantized control systems. We have provided a characterization of the reachable set for an important if particular nonlinear nonholonomic quantized system, obtained by exact sampling of the continuous–time chained–form systems. Many open problems remain in this field, that is in our opinion among the most important and challenging for applications of embedded control systems and in several other applications. Although some problems have been shown to be hard, we believe that a reasonably complete and useful system theory of quantized control system could be built by merging modern discrete mathematics techniques with classical tools of system theory.

6 Appendix

Proof of Proposition 4 Step 1. First of all we shall prove that if $\tilde{\omega}$ is comprised of elements of C then $\forall \omega \in \Omega$ $\omega\tilde{\omega}\bar{\omega}$ itself is comprised of elements of C. By definition for $\tilde{\omega} \in \mathcal{L}$ and $\forall \omega_1 \in \Omega$, $\tilde{\omega}_1 = \omega_1\tilde{\omega}\bar{\omega}_1 \in C$. Then, clearly, $\forall \omega_2 \in \Omega$,

$$\omega_2\tilde{\omega}_1\bar{\omega}_2 = (\omega_2\omega_1)\tilde{\omega}(\omega_2\bar{\omega}_1)^{-1} \in C.$$

Further, if $\tilde{\omega}_1, \ldots, \tilde{\omega}_N$ are elements of C and $\omega \in \Omega$ then

$$\omega\tilde{\omega}_1 \cdots \tilde{\omega}_N\bar{\omega} = (\omega\tilde{\omega}_1\bar{\omega}) \cdots (\omega\tilde{\omega}_i\bar{\omega}) \cdots (\omega\tilde{\omega}_N\bar{\omega})$$

is comprised of elements in C.

Step 2. Next we will show that if $\omega_1, \omega_2 \in \Omega$ then $\omega_1\omega_2\bar{\omega}_1\bar{\omega}_2$ belongs to the group generated by C. We shall see it by induction.

a) First we show that for any $w_1, w_2 \in U$ $w_1w_2\bar{w}_1\bar{w}_2$ belongs to the group generated by C. Indeed let ω s.t. $w_1w_2\omega \in \mathcal{L}$ then $w_1w_2\bar{w}_1\bar{w}_2 = (w_1w_2\omega)(\bar{\omega}\bar{w}_1\bar{w}_2)$ is the product of two elements of $\mathcal{L}$.

b) Next step is to see that if $w_1 \in U$ and $\omega_2 \in \Omega$ then property $(*)$

$(*)$ $w_1\omega_2\bar{w}_1\bar{\omega}_2$
belongs to the group generated by C.

holds true. The proof follows by induction on the length of ω_2. For length$(\omega_2) = 1$ property $(*)$ has been shown in **a)**. Suppose that we have proved $(*)$ for all ω_2 with length strictly less than N. Suppose now that length of ω_2 is equal to N.

Let $\omega_2 = w_2\omega_2'$ and ω s.t. $w_1w_2\omega \in \mathcal{L}$ then

$$\begin{aligned}&w_1\omega_2\bar{w}_1\bar{\omega}_2 =\\ &(w_1w_2\omega)\bar{\omega}\omega_2'\bar{w}_1\bar{\omega}_2 =\\ &(w_1w_2\omega)\bar{\omega}\omega_2'\bar{w}_1\bar{\omega}_2'\bar{w}_2 =\\ &(w_1w_2\omega)\bar{\omega}\omega_2'\bar{w}_1\bar{\omega}_2'w_1\omega(\bar{\omega}\bar{w}_1\bar{w}_2)\end{aligned}$$

Observe the element which lies between the two elements of C: $\bar{\omega}(\omega_2'\bar{w}_1\bar{\omega}_2'w_1)\omega$; it belongs to the group generated by C if and only if $\omega_2'\bar{w}_1\bar{\omega}_2'w_1$ belongs to the group generated by C. But now ω_2' has length less than N. Then we conclude by induction.

c) Finally $\omega_1, \omega_2 \in \Omega$ then property $(**)$

$(**)$ $\omega_1\omega_2\bar{\omega}_1\bar{\omega}_2$
belongs to the group generated by C.

holds true Again we shall prove it by induction on the length of ω_1. If length$(\omega_1) = 1$ recall the proof in **b)**. Suppose that we have proved $(**)$ for all ω_1 with length strictly less than N. Suppose now that length of ω_1 is equal to N. Let $\omega_1 = \omega_1'w_1$

$$\begin{aligned}&\omega_1\omega_2\bar{\omega}_1\bar{\omega}_2 =\\ &\omega_1'(w_1\omega_2)\bar{w}_1\bar{\omega}_1'\bar{\omega}_2 =\\ &\omega_1'(w_1\omega_2\bar{w}_1\bar{\omega}_2)\omega_2\bar{\omega}_1'\bar{\omega}_2 =\\ &\omega_1'(w_1\omega_2\bar{w}_1\bar{\omega}_2)(\omega_2\bar{\omega}_1'\bar{\omega}_2\omega_1')\bar{\omega}_1'\end{aligned}$$

The two terms in the parenthesis are elements of the group generated by C (by induction). Then the proof of **Step 2.** is completed.

Step 3. $\forall \omega \in \Omega$ and $\omega' \in \Omega$ with $\omega \equiv \omega'$ there exists some g belonging to the group generated by C such that $\omega = g\omega'$. In other words $\omega = \omega' (\mathrm{mod}\ C)$. By induction.

a. $\omega = w_1w_2$ then $(w_1w_2\bar{w}_1\bar{w}_2)w_2w_1$ with $(w_1w_2\bar{w}_1\bar{w}_2)$ an element of the group generated by C.

b. $\omega = w_1\mathbf{g}w_2$ with $\mathbf{g} = w_3w_4\bar{w}_3\bar{w}_4 \in C$ then $\omega = w_1w_2(\mathrm{mod}\ C)$.

$$\begin{aligned}&w_1(w_3w_4\bar{w}_3\bar{w}_4)w_2 =\\ &(w_1w_3\bar{w}_1\bar{w}_3)w_3(w_1w_4\bar{w}_1\bar{w}_4)w_4\\ &(w_1\bar{w}_3\bar{w}_1w_3)\bar{w}_3(w_1\bar{w}_4\bar{w}_1w_4)\bar{w}_4w_1w_2\end{aligned}$$

Let $[u, v]$ denote the commutator $uv\bar{u}\bar{v}$. For completing the proof we should prove that

$$[w_1, w_3]w_3[w_1, w_4]w_4[w_1, \bar{w}_3]\bar{w}_3[w_1, \bar{w}_4]\bar{w}_4 \in C$$

$$\begin{aligned}&[w_1, w_3]w_3[w_1, w_4]w_4[w_1, \bar{w}_3]\bar{w}_3[w_1, \bar{w}_4]\bar{w}_4 =\\ &[w_1, w_3]w_3[w_1, w_4]w_4[w_1, \bar{w}_3]\bar{w}_3\bar{w}_4\\ &(w_4[w_1, \bar{w}_4]\bar{w}_4) =\\ &[w_1, w_3]w_3[w_1, w_4]w_4\bar{w}_3\bar{w}_4\\ &(w_4w_3[w_1, \bar{w}_3]\bar{w}_3\bar{w}_4)(w_4[w_1, \bar{w}_4]\bar{w}_4) =\\ &[w_1, w_3]w_3w_4\bar{w}_3\bar{w}_4(w_4w_3\bar{w}_4[w_1, w_4]w_4\bar{w}_3\bar{w}_4)\\ &(w_4w_3[w_1, \bar{w}_3]\bar{w}_3\bar{w}_4)(w_4[w_1, \bar{w}_4]\bar{w}_4) =\\ &[w_1, w_3][w_3w_4](w_4w_3\bar{w}_4[w_1, w_4]w_4\bar{w}_3\bar{w}_4)\\ &(w_4w_3[w_1, \bar{w}_3]\bar{w}_3\bar{w}_4)(w_4[w_1, \bar{w}_4]\bar{w}_4)\end{aligned}$$

which is comprised of elements of C for what we have seen in **Step 1.**.

c. $\omega = w_1\mathbf{g}w_2$ with $\mathbf{g} = \omega[w_3w_4]\bar{\omega} \in \Omega$ then $\omega = w_1w_2 (\mathrm{mod}\ C)$. Suppose first that length$(\omega) = 1$ then

$$\begin{aligned}&w_1\omega[w_3w_4]\bar{\omega}w_2 = [w_1\omega]\omega w_1[w_3w_4]\bar{\omega}w_2 = \\ &[w_1\omega](\omega w_1[w_3w_4]\bar{w}_1\bar{\omega})\omega w_1\bar{\omega}w_2 = \\ &[w_1\omega](\omega\mathbf{w}_1[w_3w_4]\bar{w}_1\bar{\omega})[\omega w_1]w_1\omega\bar{\omega}w_2 = \\ &[w_1\omega](\omega w_1[w_3w_4]\bar{w}_1\bar{\omega})[\omega w_1]w_1w_2\end{aligned}$$

Next suppose that for all $\omega = w_1\mathbf{g}w_2$ with $\mathbf{g} = \omega[w_3w_4]\bar{\omega} \in \Omega$ with length$(\omega) < K$, it holds $\omega = w_1w_2 (\mathrm{mod}\ C)$. We shall prove it also for length$(\omega) = K$. Let $\omega = w\omega'$ then

$$\begin{aligned}&w_1\omega[w_3w_4]\bar{\omega}w_2 = w_1w\omega'[w_3w_4]\bar{\omega}'\bar{w}w_2 = \\ &[w_1w]w(w_1\omega'[w_3w_4]\bar{\omega}')\bar{w}w_2\end{aligned}$$

By the inductive hypothesis (length$(\omega') < K$) one has:

$$\begin{aligned}&w_1\omega[w_3w_4]\bar{\omega}w_2 = [w_1w]w\mathbf{g}'w_1\bar{w}w_2 = \\ &[w_1w](w\mathbf{g}'\bar{w})ww_1\bar{w}w_2\end{aligned}$$

with $\mathbf{g}'$ comprised of elements of C. Finally

$$\begin{aligned}&w_1\omega[w_3w_4]\bar{\omega}w_2 = [w_1w](w\mathbf{g}'\bar{w})[ww_1]w_1w\bar{w}w_2 = \\ &[w_1w](w\mathbf{g}'\bar{w})[ww_1]w_1w_2\end{aligned}$$

and the proof is completed.

d. Let $\omega = w_1 \ldots w_N$. Clearly by permuting the elements two by two any permutation of ω can be produced. Suppose the elements w_iw_{i+1} are permuted then, by letting $\omega_1 = w_1 \cdots w_{i-1}$ and $\omega_2 = w_{i+2} \cdots w_N$, one has

$$\omega_1w_iw_{i+1}\omega_2 = \omega_1[w_iw_{i+1}]w_{i+1}w_i\omega_2$$

If length$(\omega_1) = 1$ then by **c.** there exist some $\mathbf{g}$ comprised of elements of C either of type $[\cdot,\cdot]$ or of type $\omega[\cdot,\cdot]\bar{\omega}$ with $\omega \in \Omega$ such that

$$\omega_1[w_iw_{i+1}]w_{i+1}w_i\omega_2 = \mathbf{g}\omega_1w_{i+1}w_i\omega_2$$

Suppose that for length$(\omega_1) < K$ there exists some concatenation of elements of C, $\mathbf{g}$ either of type $[\cdot,\cdot]$ or of type $\omega[\cdot,\cdot]\bar{\omega}$ with $\omega \in \Omega$ such that

$$\omega_1[w_iw_{i+1}]w_{i+1}w_i\omega_2 = \mathbf{g}\omega_1w_{i+1}w_i\omega_2$$

Let now length$(\omega_1) = K$ and $\omega_1 = w_1\omega_1'$. Then

$$w_1\omega_1'[w_iw_{i+1}]w_{i+1}w_i\omega_2 = w_1\mathbf{g}\omega_1'w_{i+1}w_i\omega_2$$

Now $\mathbf{g}$ is comprised of elements of type $[\cdot,\cdot]$ and of type of type $\omega[\cdot,\cdot]\bar{\omega}$. We shall then use **b.** and **c.** to complete the proof.

Observe that if $\Sigma(\omega) = 0$ then $\omega = 0 (\mathrm{mod}\ C)$. Indeed if $\Sigma(\omega) = 0$ then $\omega \equiv 0$.

Step 4. We shall now proof the proposition in the general case.

Let $\hat{U} = [w^1 \cdots w^c\bar{w}^1 \cdots \bar{w}^c]$ and $N_{\hat{U}}$ the $2c \times (2c-m)$–matrix with coefficients in $\mathbb{N} \cup \{0\}$ such that $\hat{U}N_{\hat{U}} = 0$ and $\forall j = 1, \ldots, 2c-m$, $G.C.D.\{(N_{\hat{U}})_{ij}, i = 1, \ldots, 2c\} = 1$. Up to re-ordering the columns of $N_{\hat{U}}$ we can suppose that $N_{\hat{U}} = [N'_{\hat{U}}N''_{\hat{U}}]$ where $N'_{\hat{U}}$ is $2c \times (c-m)$ and $N''_{\hat{U}}$ is $2c \times c$ is comprised of the trivial columns of $N_{\hat{U}}$, i.e. for all $j = 1, \ldots, c$

$$(N''_{\hat{U}})_{ij} = \begin{cases} 1 & if\ i = j \text{ or } i - c = j \\ 0 & \text{otherwise}\end{cases}$$

Then, $\forall\omega \in \tilde{\Omega}$, one can write $\hat{\Sigma}(\omega) = N'_{\hat{U}}\alpha_1 + N''_{\hat{U}}\alpha_2$, for some $\alpha_1 \in \mathbb{N}^{c-m}, \alpha_2 \in \mathbb{N}^c$. Next we will show that $\Sigma(\omega) = N_U\alpha_1$. Indeed $\hat{U} = [U, -U]$ and

$$0 = \hat{U}N_{\hat{U}} = [U, -U]\left[N'_{\hat{U}}N''_{\hat{U}}\right] =$$

$$[U, -U]\begin{bmatrix} (N'_{\hat{U}})_+ & (N''_{\hat{U}})_+ \\ (N'_{\hat{U}})_- & (N''_{\hat{U}})_- \end{bmatrix} =$$

$$\left[U\left((N'_{\hat{U}})_+ - (N'_{\hat{U}})_-\right), U\left((N''_{\hat{U}})_+ - (N''_{\hat{U}})_-\right)\right] =$$

$$\left[U\left((N'_{\hat{U}})_+ - (N'_{\hat{U}})_-\right), U \cdot 0\right]$$

Then $N_U = \left((N'_{\hat{U}})_+ - (N'_{\hat{U}})_-\right)$ Moreover,

$$N_{\hat{U}}\begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} = \begin{bmatrix}(N'_{\hat{U}})_+\alpha_1 + (N''_{\hat{U}})_+\alpha_2 \\ (N'_{\hat{U}})_-\alpha_1 + (N''_{\hat{U}})_-\alpha_2\end{bmatrix}.$$

Thus, using the definition of $\Sigma(\omega)$ one has, for $\omega \in \tilde{\Omega}$

$$\begin{aligned}&\Sigma(\omega) = \\ &\left((N'_{\hat{U}})_+ - (N'_{\hat{U}})_-\right)\alpha_1 + \left((N''_{\hat{U}})_+ - (N''_{\hat{U}})_-\right)\alpha_2 = \\ &N_U\alpha_1\end{aligned}$$

We have seen in **Step 3.** that it is possible to permute (mod elements of C) the symbols of any word $\omega \in \Omega$. Then, if $\Sigma(\omega) = N_U\alpha_1$, a permutation can be found such that $\omega \equiv \omega_0\omega_1$ with ω_0 a concatenation of elements of $w_{0i} \in C$ of type $\omega[\cdot,\cdot]\bar{\omega}$ with

$$\begin{aligned}&\hat{\Sigma}(\omega_0) = \textstyle\sum_i \hat{\Sigma}(\omega_{0i}) = N_{\hat{U}}\begin{bmatrix}\sum_i \alpha_{0i} \\ \sum_i \alpha'_{0i}\end{bmatrix} = N_{\hat{U}}\begin{bmatrix}0 \\ \alpha'_0\end{bmatrix} \\ &\hat{\Sigma}(\omega_1) = N_{\hat{U}}\begin{bmatrix}\alpha_1 \\ \alpha'_1\end{bmatrix}\end{aligned}$$

i.e. $\Sigma(\omega_1) = N_U\alpha_1$ and $\Sigma(\omega_0) = 0$. Observe that α_1' correspond to the transit element applied to elements of $\mathcal{L}$. By the result of **Step 2.**, $\omega_1\omega_2$ is comprised of elements of C. The proof is completed.

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