Receding–Horizon Control of LTI Systems with Quantized Inputs*

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Abstract

This paper deals with the stabilization problem for a particular class of hybrid systems, namely discrete–time linear systems subject to a uniform (a priori fixed) quantization of the control set. Results of our previous work on the subject provided a description of minimal (in a specific sense) invariant sets that could be rendered maximally attractive under any quantized feedback strategy. In this paper, we consider the design of stabilizing laws that optimize a given cost index on the state and input evolution on a finite, receding horizon. Application of Model Predictive Control techniques for the solution of similar hybrid control problems through Mixed Logical Dynamical reformulations can provide a stabilizing control law, provided that the feasibility hypotheses are met. In this paper, we discuss precisely what are the shortest horizon length and the minimal invariant terminal set for which it can be guaranteed a stabilizing MPC scheme. The final paper will provide an example and simulations of the application of the control scheme to a practical quantized control problem.

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1 Introduction

Practical applications of control theory reveal some limits of the continuous models in the description of dynamical systems: limited resources or technical constraints, which finally lead to discrete measurements or to a finite number of possible control actions, are typical situations that must be faced. This is part of a broader phenomenon which is referred to as *quantization*.

In the past twenty years the problem of dynamic systems analysis and control synthesis in presence of quantization has developed and is currently growing in interest. It is now consolidated the idea of regarding quantization not as a phenomenon to be neglected and related to the concept of approximation but rather as a useful tool to be studied within proper models (see for instance [5, 11, 12, 18, 23, 24]). Many papers addressed the problem of the stabilization of quantized systems (see [7, 11, 12, 13, 15, 16, 18, 17, 25]): in [11] Delchamps clarifies that the classical concept of stability is not significant in this context, hence "practical" stability properties are introduced for quantized systems.

In the present paper we investigate the possibility of using Model Predictive Control techniques (MPC) for stabilizing the particular class of hybrid systems comprised of discrete–time linear systems subject to a uniform (a priori fixed) quantization of the control set. This approach is justified by the fact that MPC is a control policy particularly suited to cope with constrained systems, which is how quantization of the control space can be interpreted. Moreover, we take advantage of the fact that, since a predictive controller generates the control action by solving an optimization problem, the design of the controller

is reduced to a mathematical programming problem. The latter is in general a more treatable task than the computation of an explicit control law, especially when dealing with severe constraints such as input quantization and bounds on state evolution.
When applicable, the model predictive approach is successful allowing to reduce the stabilization problem to the search of invariant sets and to the study of their reachability. A detailed study of the reachability is needed to characterize the feasibility region of the optimization problem defining the model predictive controller.

Although invariant sets are very important in control theory, in the current literature [6] few results exist for quantized systems. In this paper we present some results concerning the construction of invariant sets for single–input linear systems: in this case it is possible to face the problem without relying on conservative Lyapunov techniques. The use of direct geometric considerations allows us to get results more suited to the practical implementation of MPC and to lead a quantitative analysis of the reachability problem.

The paper is organized as follows: in Section 2 it is introduced the basic MPC scheme for quantized input systems. To get stabilization results it is necessary to add suitable constraints to the basic scheme: stabilizing MPC strategies fitting for quantized systems are developed in Section 3. Questions concerning the construction of invariant sets and the feasibility of the optimization problem defining the model predictive controller are faced in Section 4. In the final Section 5 is shown how to render effective the theory developed and implement MPC. In the final version there will be also an example and simulations of the application of the control scheme to a practical quantized control problem.

Notation: $Q_n(\Lambda) := \big[-\frac{\Lambda}{2} ; \frac{\Lambda}{2} \big]^n$ is the hypercube of edge length Λ, $\lfloor x \rfloor := \max\, \{z \in \mathbb{Z} \,|\, z \leq x\}$ is the floor function, x^+ is the standard notation for $x(t+1)$, $\big(x(t)\big)_i$ stands for the i^{th} component of the state x at time t, $\|x\|_\infty := \max_{i=1,\dots,n} \big\{ \big|(x)_i\big| \big\}$, Ω^c denotes the complementary of the set Ω and ${}^t x$, ${}^t A$ are the transpose of the vector x and of the matrix A respectively.

2 Quantized Model Predictive Control (QMPC)

Consider the following linear discrete time invariant system

$$x(t+1) = Ax(t) + Bu(t) \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathcal{U} := \{u_1\,, u_2\,, \dots u_N\} \subset \mathbb{R}^m$ are the levels of quantization, and $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ is a stabilizable pair.

Consider the model predictive controller defined as

$$u(t) = u_0^*\big(x(t)\big)\,, \tag{2}$$

where, given a finite number of steps $H_p > 0$, $u_0^*\big(x(t)\big)$ is the first element of the minimizer $U^*\big(x(t)\big) \in \mathcal{U}^{H_p}$ of the following optimization problem:

$$\min_{U \in \mathcal{U}^{H_p}} \big\{ J\big(U, x(t)\big) \big\} \qquad \text{where} \tag{3a}$$

$$J\big(U, x(t)\big) = {}^t x_{H_p} P x_{H_p} + \sum_{k=0}^{H_p-1} \big({}^t x_k Q x_k + {}^t u_k R u_k \big) \tag{3b}$$

$$\text{s.t.} \begin{cases} x_0 := x(t) \\ x_{k+1} := Ax_k + Bu_k\,, \quad k = 0, \dots, H_p - 1, \\ u_k \in \mathcal{U}\,, \end{cases} \tag{3c}$$

where $R = {}^t R > 0$, $Q = {}^t Q \geq 0$, $P = {}^t P \geq 0$ are matrices of suitable dimensions. Note that in (3) x_0 is the current state, $x_1, \dots, x_{H_p}$ are the predicted states for the future H_p sampling instants when the sequence of controls $U := \big(u_0\,, u_1\,, \dots u_{H_p-1}\big) \in \mathcal{U}^{H_p}$ is applied. The minimizer (which exists because $\#\mathcal{U} < +\infty$) is denoted by $U^* := \big(u_0^*\,, u_1^*\,, \dots u_{H_p-1}^*\big)$ and the dependence on $x(t)$ is omitted for simplicity.

Although the cost function (3b) penalizes the trajectories getting far from the equilibrium, the model predictive controller does not guarantee stability (even for unconstrained stable systems) if suitable conditions on the final state are not imposed. Indeed, consider the following well known counter-example:

Example 1 Consider the discrete–time linear system

$$\begin{cases} x(t+1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \\ (x,u) \in \mathbb{R}^2_x \times \mathbb{R}_u \, . \end{cases}$$

Note that if the system is autonomous (i.e. no control is applied), then $\forall\, x(t) \in \mathbb{R}^2$, $x(t+2) = 0$.
Fix $H_p = 2$ and let

$$P = 0\,, \quad Q = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} > 0\,, \quad R = \left(\frac{1}{2}\right) .$$

Set $x(t) = \begin{pmatrix} x_0^{(1)} \\ x_0^{(2)} \end{pmatrix}$ and $U = (u_0, u_1)$, the minimizer is $U^* = \left(-\frac{6}{5} x_0^{(1)}, 0 \right)$. The model predictive controller is then $u(t) = -\frac{6}{5} x_0^{(1)}$ and induces the closed–loop dynamics

$$x(t+1) = \begin{pmatrix} -\frac{6}{5} & 0 \\ 1 & 0 \end{pmatrix} x(t)$$

which is unstable. ♣

The reason for this undesirable behaviour is that the cost function (3b) does not properly weigh, nor it imposes a constraint on the final state. While this problem could easily be solved in this simple example, the problem is more difficult when quantization is involved. The next Section develops stabilizing MPC strategies for quantized–input systems.

3 Stabilizing QMPC schemes

A suitable terminal cost P is sufficient to guarantee the closed–loop stability of system (1) when A is an asymptotically stable matrix:

Proposition 1 *Assume $0 \in \mathcal{U}$, let A be asymptotically stable and let P be the solution of the Lyapunov equation $P = {}^tAPA + Q$. Then the origin of the closed-loop system (1), (2) is globally asymptotically stable.*

Proof. Let $V\big(x(t)\big) = J\big(U^*, x(t)\big)$ be the minimum of (3) attained at time t , and denote by x_k^* the corresponding optimal trajectory. At time $t+1$, the sequence $U_1 := \big(u_1^*, \ldots\ u_{H_p-1}^*, 0\big) \in \mathcal{U}^{H_p}$ is such that the predicted states $x_k = x_{k+1}^*$, $\forall\, k = 0, \ldots, H_p - 1$, with $x(t+1) = Ax(t) + Bu_0^*$. Hence,

$$\begin{aligned} &V\big(x(t+1)\big) \le J\big(U_1, x(t+1)\big) = {}^t x_{H_p} P x_{H_p} + \\ &+ \sum_{k=0}^{H_p-1} \left\{ {}^t x_k Q x_k + {}^t u_k R u_k \right\} = {}^t(Ax_{H_p}^*) P (Ax_{H_p}^*) + \\ &+ {}^t x_{H_p}^* Q x_{H_p}^* + \sum_{k=1}^{H_p-1} \left\{ {}^t x_k^* Q x_k^* + {}^t u_k^* R u_k^* \right\} = \\ &= {}^t x_{H_p}^* P x_{H_p}^* + \sum_{k=1}^{H_p-1} \left\{ {}^t x_k^* Q x_k^* + {}^t u_k^* R u_k^* \right\} = \\ &= V\big(x(t)\big) - {}^t x(t) Q x(t) - {}^t u(t) R u(t) . \end{aligned}$$

Thus, $V\big(x(t)\big)$ is a nonnegative and non–increasing sequence, therefore it converges to a finite limit as $t \to \infty$. Hence,

$$0 \le {}^t x(t) Q x(t) + {}^t u(t) R u(t) \le V\big(x(t)\big) - V\big(x(t+1)\big) ,$$

which implies that $\lim_{t\to\infty} u(t) = 0$ because R is positive definite. Since $\mathcal{U}$ is discrete (actually it is a finite set), there exists a finite time t_0 such that $u(t) \equiv 0$ for all $t \ge t_0$: as A is asymptotically stable it follows that $\lim_{t\to\infty} x(t) = 0$. ■

Let us handle now the case of unstable systems. The quantization of the control set is a severe constraint which renders unpracticable most of the classical approaches to the stabilization problem.
The argument of [9, 22, 4], consisting of choosing P as the solution to the Riccati equation, can not be applied when the input is quantized, as the method to achieve stability is based on the assumption that the sequence $U_1 := \big(u_1^*, \ldots\ u_{H_p-1}^*, (Kx_{H_p}^*)\big)$ is an admissible one, while here in general $U_1 \notin \mathcal{U}^{H_p}$ because $Kx_{H_p}^* \notin \mathcal{U}$. On the other hand, an infinite horizon setup would be numerically intractable, as (3) would become a combinatorial problem with an infinite number of variables. The approach of using a terminal constraint $x_{H_p} = 0$, as for hybrid systems

in [3], has the drawback of having a set of initial conditions which in general are limited to a zero–measure set.

Our idea is to use a relaxed final constraint so that the feasibility of the optimization problem is achieved for a significant set of initial conditions. The terminal equality constraint is then replaced by the requirement that the initial state can be steered in H_p steps inside a *controlled–invariant* neighbourhood Ω of the equilibrium (see next Definition 1). This policy gives rise to a quantized–input version of the so–called *dual–mode* predictive control scheme (see Michalska and Mayne [14]): since the system is unstable and the control set is quantized, only *practical stability* can be achieved, on the other hand this policy has the advantage of obtaining an MPC scheme which is robust against perturbations.

We introduce now the definitions needed for the subsequent treatment.

Definition 1 The set $\Omega \subseteq \mathbb{R}^n$ is said to be *controlled–invariant* for system (1) iff $\forall\, x \in \Omega\ \exists\, u \in \mathcal{U}$ such that $x^+ = Ax + Bu\ \in \Omega$.

Definition 2 Let $\Omega \subseteq X_0 \subseteq \mathbb{R}^n$; the set Ω is said to be X_0*–attractive* iff $\forall\, x \in X_0$ there exists a trajectory which lies within X_0 and enters Ω in a finite number of steps. If moreover $\forall\, x \in X_0$ such trajectory can be chosen of length H_p, then the set Ω is said to be X_0*–attractive in* H_p *steps.*

Let $\Omega \subset \mathbb{R}^n$ be a bounded and controlled–invariant set for system (1) containing 0 in its interior part. Hence we can assume that a feedback law

$$F_\Omega : \Omega \longrightarrow \mathcal{U}$$

has been defined so that $\forall\, x \in \Omega\,,\ \ Ax + BF_\Omega(x) \in \Omega$.

We use a predictive control strategy in Ω^c to steer the states inside Ω, then we switch to the feedback law F_Ω. To this aim a slight modification of the cost function (3b) is sufficient: let

$$L(x,u) = I_{\Omega^c}(x) \cdot \big({}^t x Q x + {}^t u R u\big) \tag{4}$$

(where I_{Ω^c} is the characteristic function of Ω^c) and consider the MPC control law (2) based on the optimal control problem

$$\min_{U \in \mathcal{U}^{H_p}} \left\{ J\big(U, x(t)\big) = \sum_{k=0}^{H_p-1} L(x_k, u_k) \right\} \tag{5a}$$

$$\text{s.t.} \begin{cases} x_{H_p} \in \Omega \\ x_0 := x(t) \\ x_{k+1} := Ax_k + Bu_k\,, \quad k = 0, \dots, H_p - 1 \\ u_k \in \mathcal{U}\,. \end{cases} \tag{5b}$$

Let

$$X_{(H_p)} := \big\{ x \in \mathbb{R}^n \,|\, \exists\, U \in \mathcal{U}^{H_p} \text{ such that } x_{H_p} \in \Omega \big\}$$

be the feasibility region for the optimization problem (5): note that $\Omega \subseteq X_{(H_p)}$.

Consider the feedback law

$$F : X_{(H_p)} \longrightarrow \mathcal{U}$$

defined by

$$F(x) := \begin{cases} F_\Omega(x) & \text{if } x \in \Omega \\ u_0^* & \text{otherwise}\,, \end{cases} \tag{6}$$

where u_0^* is the first element of the minimizer U^* related to the optimization problem (5).

Note that the optimization problem (5) is solvable (i.e. the minimum is attained) for all $x(t) \in X_{(H_p)}$ because $\#\mathcal{U} < +\infty$.

Proposition 2 *Assume* Ω *is a controlled–invariant neighbourhood of the origin, and* $Q > 0$*. Then* Ω *is* $X_{(H_p)}$*–attractive for the closed–loop system (1), (6).*

Proof. The proof follows arguments similar to the ones used in the classical dual–mode MPC scheme (see for instance [8, 21]).

If $x(t) \in \Omega$ then the statement is trivial; let us prove the theorem for $x(t) \in X_{(H_p)} \setminus \Omega$. From the controlled–invariance of Ω it follows that $x \in X_{(H_p)} \Rightarrow x^+ = Ax + BF(x) \in X_{(H_p)}$, thus the trajectory starting from $x(t)$ and generated by the closed–loop dynamics (1), (6) is well defined for all sampling

instants greater than t.
With the same notations used in the proof of Proposition 1, let $U^*\big(x(t)\big) = \big(u_0^*\,,\, u_1^*\,,\, \dots\, u_{H_p-1}^*\big)$ be the minimizer of problem (5), hence at time $t+1$ the control sequence $U_1 := \big(u_1^*\,,\, \dots\, u_{H_p-1}^*\,,\, F_\Omega(x_{H_p}^*)\big) \in \mathcal{U}^{H_p}$ is such that the terminal constraint $x_{H_p} \in \Omega$ is satisfied. Thus

$V\big(x(t+1)\big) \;\leq\; J\big(U_1, x(t+1)\big) = \sum_{k=1}^{H_p-1} L(x_k^*, u_k^*) + L\big(x_{H_p}^*, F_\Omega(x_{H_p}^*)\big) = V\big(x(t)\big) - L\big(x(t), u(t)\big)$.

Since $Q > 0$ and $0 \in \mathcal{I}nt(\Omega)$, then there exists $\alpha > 0$ such that $L(x,u) \geq \alpha \quad \forall x \in \Omega^c$. Therefore

$V\big(x(t)\big) - L\big(x(t), u(t)\big) \leq V\big(x(t)\big) - \alpha\,,$

and finally

$$V\big(x(t)\big) - V\big(x(t+1)\big) \geq \alpha\,.$$

This means that as long as the trajectory is in Ω^c, the value function $V(x)$ is decreasing at least by the constant $\alpha > 0$, since $V(x) \geq 0$ this implies the thesis. ■

Remark 1 [QMPC for nonlinear systems] The linearity of system (1) has not been involved, indeed a more general result can be proved with some modifications of the arguments used to prove Proposition 2: let Ω be a controlled–invariant neighbourhood of the equilibrium state (which can be taken to be the origin) for the discrete time system

$$x(t+1) = g\big(x(t), u(t)\big)\,, \quad x \in \mathbb{R}^n,\; u \in \mathcal{U} \subset \mathbb{R}^m,$$

where $\mathcal{U}$ is a discrete control set. Consider the optimization problem (5) where in Equations (5b) $x_{k+1} := g(x_k, u_k)$ and the cost function (4) is replaced by $L(x,u) = I_{\Omega^c} \cdot \mathcal{C}(x,u)$, where $\mathcal{C}(x,u)$ is such that there exists a norm on $\mathbb{R}^n \times \mathbb{R}^m$ such that $\mathcal{C}(x,u) \geq \|(x,u)\|^2 \quad \forall (x,u) \in \mathbb{R}^n \times \mathcal{U}$. Then Ω is $X_{(H_p)}$–attractive for the closed–loop dynamics $x(t+1) = g\Big(x(t), F\big(x(t)\big)\Big)$ induced by the feedback law (6).
For the details of the proof see [16].
It is worth noting that the only hypothesis involving the function g is the assumption that Ω is a controlled–invariant set.
Of course the linear case is interesting both because there exist many efficient algorithms to implement MPC and because in this case are available results concerning the construction of controlled–invariant sets in the quantized–input case (see next Section 4).

As a consequence of Proposition 2 and Remark 1 the model predictive approach allows to reduce the stabilization problem to the search of controlled–invariant sets and to the study of their reachability (which is indeed the study of the feasibility of the optimization problem defining the model predictive controller).

4 Invariant sets

In this Section we present some results concerning the construction of controlled–invariant sets for quantized–input linear systems. We restrict to the single–input case because in this framework the construction can be done as much as possible without relying on Lyapunov techniques, which are often too conservative. The presented technique provides results which better suit the practical implementation of MPC, because we find invariant sets with polyhedral structure and also because a quantitative analysis of the reachability problem can be done.
We examine the case of a uniformly quantized control set which is fixed a priori, more precisely it is assumed that $\mathcal{U} \subseteq \epsilon\,\mathbb{Z}$ for some $\epsilon > 0$.

Let us start considering systems such that the pair (A,B) is *reachable*: in this case a change of the coordinates allows us to work with the *controller form* associated to the pair (A,B). In the following we will refer to the following hypothesis:

H1) The pair (A,B) is reachable and the system (1) is in controller form. Let $s^n - \alpha_n s^{n-1} - \cdots - \alpha_2 s - \alpha_1$ be the characteristic polynomial of A.

The basic results about controlled–invariant sets are summarized in the following statements, for a detailed exposition we refer to [18].

Proposition 3 *Assume* **H1** *and that $\mathcal{U} = \epsilon\,\mathbb{Z}$, then*

i) $\forall \Delta \geq \epsilon\,,\ Q_n(\Delta)$ *is controlled–invariant.*

ii) $\forall H_p \geq n\,,\ \ \forall\,\Delta \geq \epsilon\,,\ Q_n(\epsilon)$ *is* $Q_n(\Delta)$*–attractive in* H_p *steps.*

Moreover $Q_n(\epsilon)$ *is a minimal controlled–invariant hypercube in the following sense:*

iii) For almost all $x \in Q_n(\epsilon)\ \exists!\ u \in \mathcal{U}$ *such that* $x^+ \in Q_n(\epsilon)\,.$ □

Hence in the unbounded control set case the invariance and attractivity study is easily completed.
In the finite control set case we consider input sets of the type

$$\mathcal{U}_k := \{-k\epsilon, \ldots, 0, \ldots, +k\epsilon\}$$

and, for a given $\Delta \geq \epsilon$, we find the condition on k ensuring the controlled–invariance of $Q_n(\Delta)$:

Proposition 4 *Assume* **H1** *and that $\mathcal{U} = \mathcal{U}_k$, then $Q_n(\Delta)$ is controlled–invariant if and only if $k \geq k_0$, where:*

$$k_0 := \begin{cases} -\Big\lfloor \frac{1}{2}\frac{\Delta}{\epsilon}\big(1 - \sum_{i=1}^{n}|\alpha_i|\,\big)\Big\rfloor & \text{if } \sum_{i=1}^{n}|\alpha_i| \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

□

Once we know that $Q_n(\gamma)$ is controlled–invariant (for $\gamma \geq \epsilon$), we wish to study its attractivity properties in a fixed number of steps. The set $X_{(H_p)}$ from which $Q_n(\gamma)$ is reachable in H_p steps can be easily characterized when the matrix A is invertible: in this case, set $R_{H_p} := [A^{H_p-1}B|\cdots|AB|B]$, a simple calculation shows that

$$X_{(H_p)} = \bigcup_{U \in \mathcal{U}^{H_p}} \{A^{-H_p}\big(Q_n(\gamma)\big) - A^{-H_p}R_{H_p}U\,\}\,.$$

Anyway, since an increasing family of sets $\big(Q_n(\gamma)\big)_{\gamma\geq\epsilon}$, which can be rendered invariant, is available, we have faced another problem: to look for conditions on $\mathcal{U}_k$ ensuring that $Q_n(\epsilon)$ is $Q_n(\Delta)$–attractive (for $\Delta > \epsilon$). Solving this problem allows to introduce an MPC scheme taking into account also constraints on the trajectories (see next Remark 2).

Proposition 5 *Assume* **H1***, let $\Delta > \epsilon > 0$ and $\mathcal{U} = \mathcal{U}_k$, fix $H_p \geq n$; $H_p = n + p - 1$ with $p \geq 1$.*

A sufficient condition in order that $Q_n(\epsilon)$ is $Q_n(\Delta)$–attractive in H_p steps is that $k \geq k_p$, with

$$k_p = -\Big\lfloor \frac{1}{2}\frac{\Delta}{\epsilon}\big(1 - \sum_{i=1}^{n}|\alpha_i|\,\big) - \frac{1}{\epsilon}\frac{\Delta-\epsilon}{2\psi_p}\Big\rfloor\,,$$

where the sequence $\{\psi_m\}_{m\in\mathbb{N}^*}$ is defined as follows:

$$\begin{cases} \psi_1 := 1 \\ \psi_m := 1 + \sum_{i=1}^{m-1}|\alpha_{n-m+i+1}|\,\psi_i \quad \text{if} \\ \qquad m \geq 2\,,\ \text{where}\ \ \alpha_j := 0 \ \ \text{if}\ \ j \leq 0\,. \end{cases}$$

Moreover, if $\alpha_i \geq 0\ \forall\,i$, then the condition is also necessary. □

Remark 2 Let $\Omega \subseteq X_0 \subset \mathbb{R}^n$ and suppose that Ω is controlled–invariant and X_0–attractive in H_p steps for system (1). Consider the feedback law (6) based on the optimization problem (5) where the state constraint $x_k \in X_0 \quad \forall\,k = 0, \ldots, H_p - 1$ is added to the constraints in Equation (5b): then Ω is X_0–attractive for the induced closed–loop system.

Let us consider now the case of a *stabilizable* pair (A, B). A change of the coordinates allows us to transform the system into the following *canonical form* for a stabilizable pair:

$$A^s = \left(\begin{array}{cc} \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n_c} \end{pmatrix} & A_2 \\ 0 & A_3 \end{array}\right) \quad B^s = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

where A_3 is an asymptotically stable matrix.
Suppose (A, B) is a stabilizable pair in the canonical form: let $\mathbb{R}^n := \mathbb{R}^{n_c} \oplus \mathbb{R}^{n_{as}}$ be the underlying decomposition of the state space.
As A_3 is an asymptotically stable matrix, then $\forall\,Q \in \mathbb{R}^{n_{as}\times n_{as}}$ such that ${}^tQ = Q > 0$, $\exists!\,P > 0$ so that ${}^tA_3PA_3 - P = -Q$.
$V(z) := {}^tzPz$ is a Lyapunov function for the system $z^+ = A_3z$, hence $\forall\,r \geq 0$, $\{z \in \mathbb{R}^{n_{as}}\,|\,V(z) \leq r\}$ is

A_3 –invariant.
Note that

$$z \in \{V \le r\} \ \Rightarrow \ \| A_2 \, z \|_\infty \le \gamma(r) \, .$$

With the notations introduced we can state the following:

Proposition 6 *Assume* $\mathcal{U} := \mathbb{Z}$, $\forall\, r \ge 0$ *the set* $\Omega_r := \Big(\big[-\frac{1}{2} - (n_c - 1)\gamma(r) \, ; \ \frac{1}{2} + (n_c - 1)\gamma(r) \big] \times \times \big[-\frac{1}{2} - (n_c - 2)\gamma(r) \, ; \ \frac{1}{2} + (n_c - 2)\gamma(r) \big] \times \cdots \times \big[-\frac{1}{2} - \gamma(r) \, ; \ \frac{1}{2} + \gamma(r) \big] \times \big[-\frac{1}{2} \, ; \ \frac{1}{2} \big] \Big) \times \{V \le r\}$ *is controlled–invariant.*

Proof. $y := \begin{pmatrix} x \\ z \end{pmatrix} \in \Omega_r$ iff $\forall\, i = 1, \ldots, n_c \quad |x_i| \le \frac{1}{2} + (n_c - i)\gamma(r)$ and $z \in \{V \le r\}$ by definition.
$y^+ = \begin{pmatrix} x^+ \\ z^+ \end{pmatrix}$ is such that
$(x^+)_1 = (x)_2 + (A_2 z)_1$ thus
$|(x^+)_1| \le \big(\frac{1}{2} + (n_c - 2)\gamma(r) \big) + \gamma(r) \, ;$
$\vdots$
$(x^+)_{n_c - 1} = (x)_{n_c} + (A_2 z)_{n_c - 1}$ thus
$|(x^+)_{n_c - 1}| \le \frac{1}{2} + \gamma(r) \, ;$
$(x^+)_{n_c} = \sum_{i=1}^{n_c} \alpha_i (x)_i + (A_2 z)_{n_c} + u$ hence we can choose $u \in \mathbb{Z}$ such that $|(x^+)_{n_c}| \le \frac{1}{2} \, ;$
$z^+ \in \{V \le r\}$. ■

The case $\mathcal{U} = \epsilon\, \mathbb{Z}$ is analogous.

Remark 3 The invariant sets found in this case can be significantly larger then the ones found in the reachable case: this because of the uncontrollable part $z^+ = A_3 z$ of the system which gives rise to the constant $\gamma(r)$ affecting the reachable part. However $\lim_{t \to \infty} z(t) = 0$, this means that in practice the states evolve inside Ω_r converging towards the set $\big([-\frac{1}{2} \, ; \ \frac{1}{2}]^{n_c} \big) \times \{0\}$.

5 QMPC Computations

It is immediate to cast problems (3) and (5) as integer quadratic programs (IQPs). Indeed, by substituting $x_k = A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{k-1-j}$, Equation (3b) can be rewritten as

$$\begin{array}{ll} \min\limits_{U} & \big\{ \frac{1}{2} U' H U + x'(t) F U + \frac{1}{2} x'(t) Y x(t) \big\} \\ \text{subj. to} & G U \le W + E x(t) \\ & u_k \in \mathcal{U}, \ \forall k = 0, \ldots, H_p - 1 \end{array} \tag{7}$$

where the column vector $U := {}^t[{}^t u_0 \, , \, \ldots \, {}^t u_{H_p - 1}] \in \mathbb{R}^{m H_p}$, is the optimization vector, $H = {}^t H \succ 0$, and H , F , Y , G , W , E are easily obtained from Q , R , and (3b) (as only the optimizer U is needed, the term involving Y is usually removed from (7)).

The optimization problem (3) is an IQP. Because the problem depends on the current state $x(t)$, the implementation of MPC requires the on-line solution of an IQP at each time step. Although efficient (mixed) integer quadratic programming solvers based on branch and bound methods are available [10, 20, 2], computing the input $u(t)$ demands significant on-line computation effort. In [1] we provide a multiparametric integer programming algorithm to compute off line the equivalent piecewise constant form of the MPC control law defined by (7).

Problem (5) can instead cast as a mixed integer quadratic program (MIQP). By defining

$$\begin{array}{rcl} z_k^x(x_k, u_k) & = & \left\{ \begin{array}{ll} x_k & \text{if } \ x_k \in \Omega \\ 0 & \text{otherwise} \end{array} \right. \\ z_k^u(x_k, u_k) & = & \left\{ \begin{array}{ll} u_k & \text{if } \ x_k \in \Omega \\ 0 & \text{otherwise} \end{array} \right. \end{array} \tag{8}$$

we get $L(x_k, u_k) = {}^t z_k^x Q z_k^x + {}^t z_k^u E z_k^u$, and by setting

$$[\delta_k = 1] \leftrightarrow [x_k \in \Omega] \tag{9}$$

we finally obtain

$$\begin{array}{ll} \min\limits_{U, Z, \Delta} & \Big\{ \sum_{k=0}^{H_p - 1} z_k^{x\prime} Q z_k^x + z_k^{u\prime} E z_k^u \Big\} \\ \text{subj. to} & z_k^x = \big(A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{k-1-j} \big) \delta_k \\ & [\delta_k = 1] \leftrightarrow [x_k \in \Omega] \\ & A^{H_p} x(t) + \sum_{j=0}^{H_p - 1} A^j B u_{H_p - 1 - j} \in \Omega \\ & u_k \in \mathcal{U}, \ \forall k = 0, \ldots, H_p - 1 \end{array} \tag{10}$$

where $Z := {}^t[{}^t z_0^x \ \ldots \ {}^t z_{H_p - 1}^x \ {}^t z_0^u \ \ldots \ {}^t z_{H_p - 1}^u]$, $\Delta := {}^t[\delta_0 \ \ldots \ \delta_{H_p - 1}]$, which can be translated to MIQP according to standard techniques (see e.g. [3]).

Remark 4 State constraints having the form $x_{\min} \leq x(t) \leq x_{\max}$ (or, more generally, $\bar{A}x(t) \leq \bar{B}$) can be easily taken into account by adding the linear constraints

$$\bar{A}\left[A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{k-1-j}\right] \leq \bar{B}$$

in (10). Clearly, in this case the invariant set Ω must be contained in the polyhedron $\bar{A}x \leq \bar{B}$ in order to guarantee constraint fulfillment of the QMPC controller at all times $t \geq 0$.
Hence the algorithm can take into account state constraints also along the predicted trajectories: this is an important extension of the recent approach presented in [19] which is not provided for dealing with state constraints.

Conclusions

In this paper we have considered the stabilization problem for discrete–time linear systems subject to a fixed, uniform quantization of control inputs. It has been shown that the Model Predictive Control approach can be profitably used and we provided conditions for its applicability. Several open problems remain in this field, among which is the extension to (quantized) output MPC. More generally, the combination of quantization with limited communication bandwidth is a most important and challenging area to which further work will be devoted.

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