On the Closure Properties of Robotic Grasping *

Antonio Bicchi

Centro "E. Piaggio"
Università di Pisa
via Diotisalvi, 2
56125 Pisa - Italia

Tel.: +39–50–565328 / 553639
Fax: +39–50–565333 / 550650
Email: bicchi@vm.cnuce.cnr.it

Abstract

The form–closure and force–closure properties of robotic grasping are investigated. Loosely speaking, these properties are related to the capability of the robot to inhibit motions of the workpiece in spite of externally applied forces. In this paper, form–closure is considered as a purely geometric property of a set of unilateral (contact) constraints, such as those applied on a workpiece by a mechanical fixture, while force–closure is related with the capability of the particular robotic device being considered to apply forces through contacts. The concepts of partial form– and force–closure properties are introduced and discussed, and an algorithm is proposed to obtain a synthetic geometric description of partial form–closure constraints. While the literature abounds with form–closure tests, proposed algorithms for testing force–closure are either approximate or computationally expensive. This paper proves the equivalence of force–closure analysis with the study of the equilibria of an ordinary differential equation, to which Lyapunov's direct method is applied. This observation leads to an efficient algorithm for assessing the force–closure property of the grasp.

*Submitted, Int. J. of Robotics Research, Nov. 1993. Accepted, Jul. 1994.

1 Introduction

A number of recent papers dealing with the analysis of grasping mechanisms, especially in robotics, underscore the fundamental role played by the so-called "form–closure" and "force–closure" properties. These properties concern the capability of the grasp to completely or partially constrain the motions of the manipulated object, and to apply arbitrary contact forces (and hence command arbitrary trajectories) on the object itself, without violating friction constraints at the contacts.

Form–closure, originally investigated by Reuleaux [1875], is related with the ability of constraining devices to prevent motions of the grasped object, relying only on unilateral, frictionless contact constraints. An example of a form–closure grasp of a square object by means of four contacts is depicted in fig.1–a. Contacts are represented with fixed pins at the contact points, indicating that only motions of the object that cause penetration of the pin in the object are prevented by that constraint. Reuleaux showed that at least four contact points are necessary to achieve the form–closure property in the planar case, and Somov [1900] found that at least seven are needed in the general spatial case. Lakshminarayana [1978] reports about these results and gives new proofs in the spatial case. The latter author also introduced the idea of partially restraining grasps, to take into account the more general case of devices intended to allow only some degrees–of–freedom to the object. An example of what will be later defined as a "partially form–closure" grasp is reported in fig.1–b, where only translation in the horizontal direction is allowed to the object by the four contact constraints. Ohwovoriole [1980] and Salisbury [1982] introduced closure properties in the robotics literature and used screw theory for approaching the problem. The analysis of form–closure is intrinsically geometric, in so far as it does not consider the kinematics of the grasping mechanism (hand or fixture), nor the magnitude of contact forces, and it lends to rather elegant treatments mostly derived from linear and convex programming techniques. Many contributions to form–closure study have been focused on the problem of grasp synthesis, i.e. given the object geometry, to place contacts so as to prevent object motions. Baker *et al.* [1985], Mishra *et al.* [1986], Selig and Rooney [1989], and Markenscoff *et al.* [1990] successively discussed the possibility of finding form–closure grasps on different surfaces, and showed that there is no such grasp on finite surfaces of revolution. With that exception, Mishra *et al.* [1986] provided an upper bound to the number of contacts necessary to synthetize form–closure grasps on arbitrary objects. The bound was later refined for a narrower but important class of objects, including all polyhedra, by Markenscoff *et al.* [1990], who showed that four contacts are also sufficient for the form–closure grasp of any planar object in that class (seven contacts are sufficient in the 3D case). Constructive procedures for placing contacts on given objects to achieve form–closure have been proposed in a series of papers originated by Nguyen [1986], and Markenscoff and Papadimitriou [1989]. The analysis problem, i.e. given an object and a set of contact locations, to decide whether the object has any degree-of-freedom left (and which), has been comparatively less thoroughly investigated so far. Qualitative (true/false) tests for form–closure have been proposed by Lakshminarayana [1978], Salisbury [1982], Nguyen [1986], Mishra *et al.* [1986], and Hirai and Asada [1993]. On the other hand, Kirkpatrick *et al.* [1989] and Trinkle [1992] proposed quantitative tests, i.e., provided a quality index for the grasp under examination. It should be noted

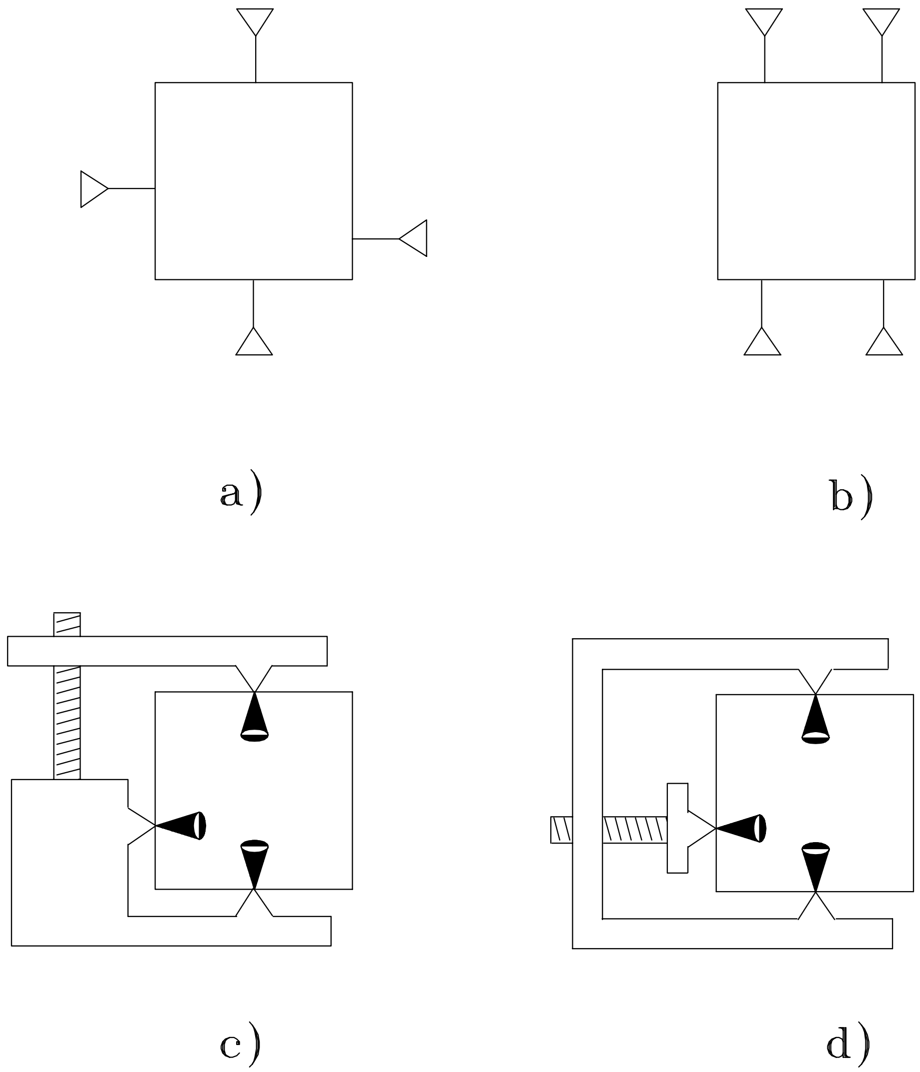


Figure 1: Four examples introducing closure definitions

that efficient form–closure tests can sometimes be derived, with minor adjustments, by techniques proposed in the closely related field of optimal force distribution in multiple chain robots (see e.g. [Cheng and Orin, 1990]).

After Lakshminarayana [1978] introduced it, the concept of partial form–closure has found application in the field of workpart fixturing design, where Asada and By [1985] proposed the related concepts of accessibility and detachability of workpieces. Brock [1988] considered partially restrained grasps from the point of view of dexterous manipulation. Trinkle [1992] uses the term "strong force–closure" to refer to what in this paper is referred to as partial form–closure, and provides a quantitative test for it. In section 2 of this paper we consider form–closure related definitions and algorithms, reviewing existing results and providing a more complete geometric characterization of partially form–closure grasps.

In the second part of this paper, the force–closure property of robotic grasping is considered. While there is a wide consensus in the literature on the definition of form–closure, the concept of force–closure is somewhat less clearcut and universally accepted. The intuitive meaning of force–closure implies that motions of the grasped object are completely (or partially) restrained despite whatever external disturbance, by virtue of suitably large contact forces that the constraining device (the end-effector) is actually capable to exert on the object.

Perhaps, the distinction between form– and force–closure that is most often made in the literature is that frictional contact forces are considered in force–closure analysis, while no friction is considered in form–closure. However, as already mentioned and as shown later in more detail, the frictional nature of contacts is inessential for 2D grasps and of limited relevance in general.

A more relevant characterization of force–closure grasping can be based on the fact that, as opposite to the purely geometric nature of form–closure, force–closure (in the intuitive meaning above) involves consideration of how contact forces can be applied on the object, and, as a consequence, the kinematics of the end-effector should play a role in force–closure. This is indeed the point of view being proposed in this paper. To explain why the end-effector kinematic structure is relevant to force–closure, consider the grasps depicted in fig.1–c and 1–d, where the same object is held by two different end-effectors through three identical contacts (friction cones are depicted by shaded sectors). It is intuitively clear that, while the grasp in fig.1–c can resist arbitrary forces externally applied on the object by suitably "squeezing" the object, the grasp in fig.1–d can not oppose e.g. to forces pulling the object to the right in the horizontal direction, since no "squeezing" is allowed by the end-effector. Similar cases may occur whenever the end-effector has fewer degrees–of–freedom than necessary to arbitrarily control contact forces (i.e., it is *kinematically defective*). Far from being a pathological case, kinematic deficiency is rather a normal condition in simple industry-oriented grippers, as well as in more complex devices such as dextrous robot hands when used in "power grasp" configuration.

The study of force–closure is of obvious importance in the choice of grasping mechanisms, with particular regard to positioning the fingers of a robotic hand on the grasped object so as to guarantee robustness against slippage. The synthesis of force–closure grasps has been considered by Nguyen [1986], [1988], who provided tools for constructing robust force–closure robotic grasps on polyhedral objects. Robustness in spite of errors in locating contacts was also a concern of Park and Starr [1992]; Ponce *et al.* [1993] extended Nguyen's methods to grasp curved 2D objects. Methods for the analysis of force–closure have been considered by Nguyen [1988] for 2-finger grasps. Ferrari and Canny [1992] proposed a quantitative test, suitable for planning optimal grasps, while Chen and Burdick [1993] proposed a qualitative test for n-finger grasps in 2D. These force–closure tests can be regarded, from the point of view adopted in this paper, as form–closure tests applied on suitably modified grasp configurations. This approach cannot be generalized to 3D grasps without introducing an approximation of friction cones by pyramids, as will be discussed in section 3 of this paper. To our knowledge, the only exact force–closure test on n-finger, 3D grasps has been described by Nakamura *et al.* [1989]. The method described therein requires the solution of 12 constrained non-linear programming problems. In the literature on grasping and closure analysis, little attention seems to have been payed to the

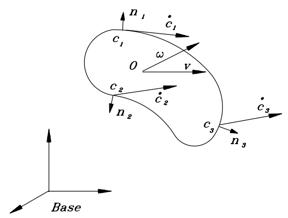


Figure 2: An object constrained by three contact points.

role of the end-effector structure and kinematics, with the notable exceptions of Trinkle *et al.* [1987], Waldron *et al.* [1989], Pollard and Lozano-Perez [1990], and Hunt *et al.*, [1991].

In section 3 of this paper we introduce a definition of force–closure that takes into account the kinematics of the gripping device. Further, we show the equivalence between the investigation of force–closure with the study of the equilibria of an ordinary differential equation, the stability analysis of which can be studied by Lyapunov's direct method. This originates a very efficient algorithm for exactly assessing the force–closure property of a robotic grasp. Further, a slight modification of the algorithm leads to the definition of a practical quality index for force–closure, that can be used for planning optimal grasps. An example of this is discussed in section 4.

2 Form–Closure

Consider a rigid object O whose infinitesimal motion in three-dimensional space is described, in a fixed base frame, by the linear velocity $\mathbf{v}$ of a reference point fixed with the object and by its angular velocity ω (see fig.2). We are interested in the properties of sets of constraints on the velocities of points $\mathbf{c}_i, \quad i = 1, \ldots, n$ on the surface of O that prevent the velocity of contact points to have components along a single direction (contact constraints). These constraints can be expressed as

$$\mathbf{n}_i^T \dot{\mathbf{c}}_i \geq 0, \tag{1}$$

where $\mathbf{n}_i \in \mathbb{R}^3$ is the unit vector along the forbidden direction and pointing into the surface at $\mathbf{c}_i$. Frictionless contacts are modeled if $\mathbf{n}_i$ is chosen normal to the surface at $\mathbf{c}_i$. The velocity (expressed in base frame) of $\mathbf{c}_i$ can be written as

$$\dot{\mathbf{c}}_i = \mathbf{v} + \omega \times \mathbf{c}_i, \tag{2}$$

Juxtaposing n such relationships we have in matrix notation

$$\dot{\mathbf{c}} = \mathbf{G}^T \dot{\mathbf{u}}, \tag{3}$$

$$\mathbf{N}^T \dot{\mathbf{c}} \geq 0 \tag{4}$$

where $\dot{\mathbf{u}} = (\mathbf{v}^T, \omega^T)^T \in \mathbb{R}^6$, $\dot{\mathbf{c}} = (\dot{\mathbf{c}}_1^T, \ldots, \dot{\mathbf{c}}_n^T)^T \in \mathbb{R}^{3n}$, $\mathbf{G}$ is the so-called grasp matrix

$$\mathbf{G} = \begin{pmatrix} \mathbf{I}_3 & \cdots & \mathbf{I}_3 \\ \mathbf{S}(\mathbf{c_1}) & \cdots & \mathbf{S}(\mathbf{c_n}) \end{pmatrix},$$

$\mathbf{S}(\mathbf{c_i})$ is the cross-product matrix for $\mathbf{c}_i$, and $\mathbf{N} =$ diag $(\mathbf{n}_1, \ldots, \mathbf{n}_n)$. Note that inequality symbols when used for vectors are meant elementwise. Analogous definitions hold for the planar case, that are omitted here. In the following, d indicates the dimension of the object configuration space, i.e. $d = 6$ in three-dimensional space, and $d = 3$ in the plane.

Definition 1 *A set of contact constraints is defined* Form–Closure *if, for all object motions* $\dot{\mathbf{u}} \in \mathbb{R}^d$*, at least one contact constraint is violated.*

Checking the form–closure property is usually regarded as a linear programming problem of the standard form

$$\begin{cases} \text{Maximize } \mathbf{f}^T\mathbf{x} \\ \text{subject to } \mathbf{N}^T\mathbf{G}^T\mathbf{x} \geq 0 \end{cases} \tag{5}$$

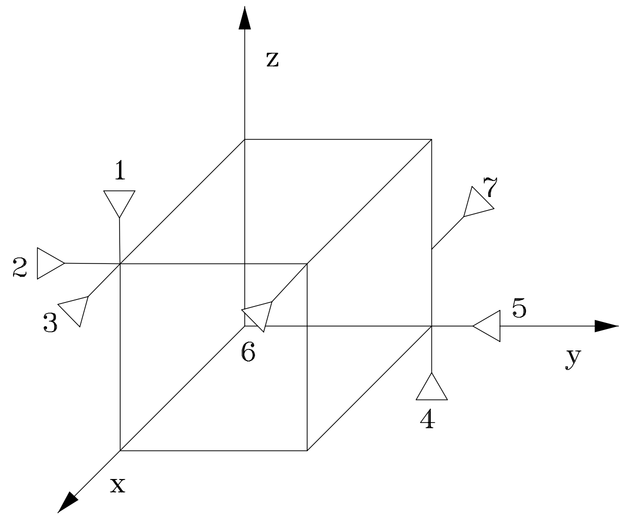
where $\mathbf{f} \in \mathbb{R}^d$ is an arbitrary constant vector. The existence of a feasible solution for any of these problems is a necessary and sufficient condition for negating the form–closure property of a set of contact constraints. Eq.(5) is sometimes given a physical interpretation, where $\mathbf{f}$ is a force/torque field (e.g., due to gravity) applied on the object, because of which the object moves if a feasible solution exists such that the mechanical work developed, $\mathbf{f}^T\mathbf{x}$, is positive. This interpretation can be used for determining the object motions (see e.g. Trinkle [1992]). Considering the standard LP problem (5) also allows application of efficient algorithms such as the compact–dual method (see [Cheng and Orin, 1990]). However, note that, strictly speaking, the concept of "force" is inessential to form–closure and can be altogether avoided in its treatment.

The Reuleaux-Somov condition on the number of contacts necessary for form–closure can be easily derived from the above formulation:

Proposition 1 (Reuleaux-Somov) . *The minimum number of contacts necessary to form-restrain an object in its configuration space is* $d + 1$.

Proof: Since matrix $\mathbf{N}^T\mathbf{G}^T$ has n rows and d columns, if $n < d$ (or $n = d$ and rank $(\mathbf{N}^T\mathbf{G}^T) < d$) there exists $\mathbf{x} \neq 0$ lying in the nullspace of $\mathbf{N}^T\mathbf{G}^T$. If otherwise $n = d$ and $\mathbf{N}^T\mathbf{G}^T$ is invertible, then a solution of $\mathbf{N}^T\mathbf{G}^T\mathbf{x} = \eta \geq 0$ can be found as $\mathbf{x} = (\mathbf{N}^T\mathbf{G}^T)^{-1}\eta$. In both cases a feasible solution of (5) exists, hence the necessary condition for a form–closure grasp: $n \geq d + 1$. □

The condition derived from the above equivalent linear programming problem differs from that proposed by Lakshminarayana [1978], who explicitly considers the case when the constraint matrix $\mathbf{GN}$ is full row rank. In that hypothesis, the above formulation can be reversed using the theorem of the separating hyperplane known from duality theory in linear programming (see e.g. [Gale, 1960]):



Contact	Point	Direction
1	[1 0 1]	[0 0 -1]
2	[1 0 1]	[0 1 0]
3	[1 0 1]	[-1 0 0]
4	[0 1 0]	[0 0 1]
5	[0 1 0]	[0 -1 0]
6	[1 1 1]	[-1 0 0]
7	[0 1 0.5]	[1 0 0]

Figure 3: A cube constrained by seven contact points.

Proposition 2 (Lakshminarayana) . *A set of contact constraints is form–closure if and only if its constraint matrix* $\mathbf{GN}$ *is full row rank and there exists* $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} > 0$, *such that* $\mathbf{GNy} = 0$.

Numerical routines are generally available in mathematical software packages that provide simple and efficient tests for either formulation of the problem (the search for a feasible solution actually constitutes the preliminary step of well-known linear programming algorithms such as the Simplex or Karmarkar's methods). Other equivalent formulations of the form–closure test have been derived and used in the literature to obtain efficient algorithms. Geometrically reformulated, proposition 2 requires that the convex hull of the columns of $\mathbf{GN}$, $CH(\mathbf{GN})$, contains a neighborhood of the origin ([Mishra *et al.*, 1986]), or, equivalently, that no plane through the origin and containing any two column vectors of $\mathbf{GN}$ is a supporting hyperplane of $CH(\mathbf{GN})$. The latter observation has been used by Chen and Burdick [1993] to derive an efficient test algorithm applicable to form–closure.

Example 1. As an example of application of the form–closure analysis, the problem of constraining a cube by using seven contacts is often reported ([Lakshminarayana, 1978]; [Nguyen, 1988]). With reference to fig.3, the constraint matrix $\mathbf{GN}$ is built as

$$\mathbf{GN} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0.5 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Since the constraint matrix is full row rank, form–closure can be easily verified using proposition 2 (e.g., $\mathbf{GNx} = 0$ for $\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$).

In many cases, even though form–closure may not be verified, contact constraints partially restrain the motions of the object. Actually, these cases are most relevant to workpiece fixturing and dextrous manipulation.

In fact, the existence of feasible motions for the workpiece in the vicinities of the designated fixturing configuration is a necessary condition for the existence of a path to bring the workpiece in the fixturing position, or to remove it after machining. To characterize this problem, Asada and By [1985] introduced the property of *accessibility and detachability.* In its basic form this property is equivalent to the negation of form–closure, but it has a "strong" form that requires that the object can be placed or removed from the grasp by moving in a nearby region where none of the contacts remain active. The analysis of these properties, and the subsequent phase of planning workpiece motions in the working cell, clearly require a geometric characterization of the set of motions allowed by constraints.

Furthermore, in many fine manipulation operations, human hands constrain in turn only some motions of the object while setting others free to slip, thus increasing the dexterity of manipulation. It can be expected that this type of manipulation could provide better dexterity and efficiency than techniques based on finger "gaiting" through successive statically stable configurations, such as those described by Tournassoud *et al.* [1987]. Fearing [1986], Brost [1988], Brock [1988], Cole *et al.* [1992], and Chen and Burdick [1993] have successively considered this problem. However, the synthesis of sets of contact locations for selectively preventing and allowing slippage motions of grasped objects seems far from being satisfactorily achieved.

In the following, it is our purpose to contribute to the partial form–closure analysis problem, by studying the subsets of object motions that are prevented and allowed, respectively, by a given set of contacts. Accordingly, we introduce the concept of partial form–closure:

Definition 2 *A set of contact constraints is defined* partially form–closure *with respect to a subset* $U \subset \mathbb{R}^d$ *if, for all object motions* $\dot{\mathbf{u}} \in U$*, at least one contact constraint is violated.*

The *constrained subset* U_c is defined as the smallest set that contains every form-restrained subset of object motions. Correspondingly, let $U_f = \{\mathbf{u} \in \mathbb{R}^d | \mathbf{N}^T \mathbf{G}^T \mathbf{u} \geq 0\}$ ($U_f \equiv \mathbb{R}^d \setminus U_c$) denote the *free subset* of object motions.

Definition 3 (Asada and By, 1985) *A set of contact constraints is defined* accessible and detachable *(A.D.) if* $U_f \neq \emptyset$*;* strongly *so (S.A.D.) if the interior of* U_f *is not void.*

In general, linear subspaces of $\mathbb{R}^d$ may be embedded both in U_c and U_f. The existence and maximal dimension of linear subspaces embedded in the free subset can be easily tested according to the following

Proposition 3 *The largest subspace* $\bar{U}_f \subset \mathbb{R}^d$ *embedded in* U_f *is the nullspace of* $\mathbf{N}^T \mathbf{G}^T$*.*

Consider $\mathbf{u}_o \in \bar{U}_f$, by definition of U_f it holds $\mathbf{N}^T \mathbf{G}^T \mathbf{u}_o \geq 0$. Since $\bar{U}_f$ is a subspace, it must also hold $-\mathbf{N}^T \mathbf{G}^T \mathbf{u}_o \geq 0$. These inequalities together imply $\mathbf{N}^T \mathbf{G}^T \mathbf{u}_o = 0$. $\square$

The Reuleaux-Somov condition 1 can be easily generalized to partial form–closure with respect to subspaces in U_c:

Proposition 4 *The minimum number of contacts necessary to partially form-restrain an object with respect to an m-dimensional subspace is $m+1$.*

Proof: Consider a constrained subspace $U \subset U_c$, and let the columns of the matrix $\mathbf{U}$ form a basis of U. Then, any $\dot{\mathbf{u}} \in U$ can be written as $\dot{\mathbf{u}} = \mathbf{U}\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^m$, and the above introduced equivalent linear programming problem can be modified for partial form–closure simply as

$$\begin{cases} \text{Maximize } \mathbf{f}^T\mathbf{x} \\ \text{subject to } \mathbf{N}^T\mathbf{G}^T\mathbf{U}x \geq 0 \end{cases} \tag{6}$$

The rest of the proof follows that of proposition 1. □

Next, the geometric structure of the free and constrained subsets of object motions in a grasp are investigated.

Proposition 5 *The free and constrained subsets of object motions are cones in $\mathbb{R}^d$; the free subset U_f is a convex polyhedral cone.*

Proof: From definition 2, it can be easily verified that $\mathbf{x} \in U_c$ implies $\mu\mathbf{x} \in U_c$ for all $\mu > 0$, hence U_c is a cone. Also, by its construction, the free subset is the intersection of the closed halfspaces defined by contact constraints: $U_f = \bigcap_{i=1,n} S_i$, $S_i = \left\{\mathbf{x} \in \mathbb{R}^d | \mathbf{n}_i^T\mathbf{G}^T\mathbf{x} \geq 0\right\}$, hence it is a polyhedral convex cone. □

Because they belong to a convex polyhedral cone, all free object motions can be described as a positive linear combination of a finite number of basis vectors. In the terminology of polyhedral convex cones, this change of description is termed *conversion from* face *form to* span *form* ([Goldman and Tucker, 1956], [Hirai and Asada, 1993]). Related algorithms are derived from linear programming techniques. An algorithmic description of a convex basis of U_f, $CB(U_f)$, i.e. of a set with minimal cardinality composed of vectors that positively span the free subset, is provided in the following.

Algorithm 1 (Basis of the Free Motion Cone) . Consider the nullspace W_i of the $i-th$ contact constraint vector $\mathbf{n}_i^T\mathbf{G}$. Suppose at first that the constraint matrix $\mathbf{GN}$ is full rank, hence the number of constraints n is greater or equal to d. Form all possible intersections of the n constraint nullspaces $d-1$ at a time: these operations will provide at least $s_0 = \begin{pmatrix} n \\ d-1 \end{pmatrix}$ vectors $\mathbf{w}_j$, $j = 1, \ldots, s$ [1]. Operate on this set of vectors as follows: if $\mathbf{N}^T\mathbf{G}^T\mathbf{w}_j \geq 0$, leave $\mathbf{w}_j$ unchanged; if $\mathbf{N}^T\mathbf{G}^T\mathbf{w}_j \leq 0$, change sign to every component of $\mathbf{w}_j$; if neither condition applies, discard $\mathbf{w}_j$. Finally, discard any vector in the set resulting from these operations that can be obtained as a convex combination of other vectors in the set (i.e., take the vertices of the convex hull of the set). The resulting set of vectors form $CB(U_f)$ (this follows from the fact that U_f is a strictly convex polyhedral cone, and the $\mathbf{w}_j$'s lie at its edges). The algorithm above is illustrated in Example 2.
If the rank of $\mathbf{GN}$ is $d-h$, $d-1 > h > 0$, then the h-dimensional nullspace W of $\mathbf{N}^T\mathbf{G}^T$ belongs to U_f. The boundary of the (not strictly) convex polyhedral cone U_f is formed by W and by $(h+1)$-dimensional half-hyperplanes all of which intersect in W. Form all possible intersections of the n constraint nullspaces W_i $d-h-1$ at a time, thus obtaining

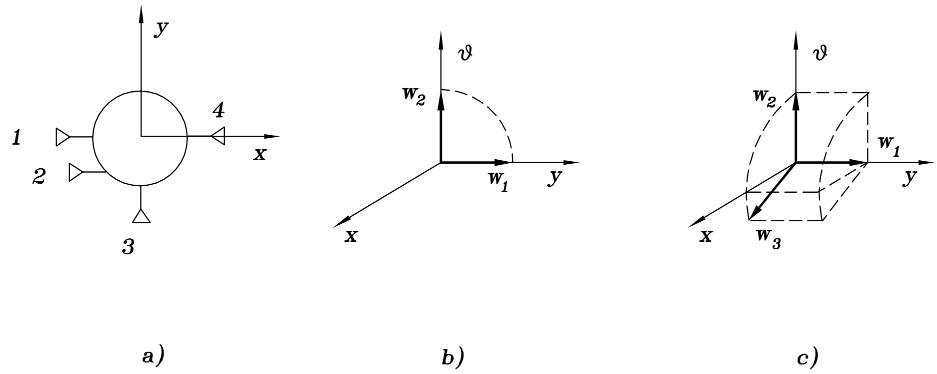


Figure 4: Example 2.

at least $(h+1)\left(\begin{array}{c} n \\ d-h-1 \end{array}\right)$ vectors spanning the supporting hyperplanes [2] . Note that each intersection results in at least $h+1$ vectors that can be linearly combined so as to have a basis of W in the first h vectors followed by vectors $\mathbf{w}_j \notin W$. Operate on the latter vectors $\mathbf{w}_j, j > h$ as described above, i.e., if $\mathbf{N}^T\mathbf{G}^T\mathbf{w}_j \geq 0$, leave $\mathbf{w}_j$ unchanged; if $\mathbf{N}^T\mathbf{G}^T\mathbf{w}_j \leq 0$, change sign to every component of $\mathbf{w}_j$; if neither condition applies, discard $\mathbf{w}_j$; hence take the vertices of the convex hull of the resulting set. Let $\mathbf{w}_k$, $k = 1, \ldots, h$ denote any set of basis vectors for W: the set of $\mathbf{w}_k$'s and $-\mathbf{w}_k$'s is a convex basis for W. Finally, a convex basis for U_f is obtained as the union of the vertices of the convex hull of the $\mathbf{w}_j$'s with the convex basis for W (see Example 3).

The above algorithm can be applied also to the limit case $h = d-1$, provided that the intersection vectors are replaced by the constraints, $\mathbf{w}_{h+j}^T = \mathbf{n}_j^T\mathbf{G}$ (see Example 4). (end of algorithm 1)

Based on the the fact that $CB(U_f)$ has minimal cardinality, we have the following

Proposition 6 *A set of contact constraints is strongly accessible and detachable only if the cardinality of $C_B(U_f)$ is greater or equal to d. Ditto if and only if the subspace spanned by vectors in $CB(U_f)$ has dimension d.*

Example 2. Consider the planar grasp of the object depicted in fig.4–a. Contacts are placed at points $\mathbf{c}_1 = [-1\,0]^T$; $\mathbf{c}_2 = [-\sqrt{2}/2 - \sqrt{2}/2]^T$; $\mathbf{c}_3 = [0\,-1]^T$; $\mathbf{c}_4 = [1\,0]^T$, and the associated directions are $\mathbf{n}_1 = [1\,0]^T$; $\mathbf{c}_2 = [1\,0]^T$; $\mathbf{c}_3 = [0\,1]^T$; $\mathbf{c}_4 = [-1\,0]^T$. Accordingly, the matrix $\mathbf{GN}$ is

$$\mathbf{GN} = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{2}/2 & 0 & 0 \end{bmatrix},$$

and is full row rank. The nullspaces of the individual contact constraints are

$$W_1 = \mathsf{span}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; W_2 = \mathsf{span}\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}; W_3 = \mathsf{span}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}; W_4 = \mathsf{span}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The intersections of these subspaces taken $(d-1=3-1=)$ 2 at a time have basis vectors $\mathbf{w}_j$ given by

$$\begin{array}{c} W_1 \cap W_2 = \mathsf{span}\ \mathbf{w}_1;\ \ \mathbf{w}_1 = [0\,1\,0]^T; \\ W_1 \cap W_3 = \mathsf{span}\ \mathbf{w}_2;\ \ \mathbf{w}_2 = [0\,0\,1]^T; \\ W_2 \cap W_3 = \mathsf{span}\ \mathbf{w}_3;\ \ \mathbf{w}_3 = [-1\,0\,\sqrt{2}]^T; \\ W_1 \cap W_4 = \mathsf{span}\ [\mathbf{w}_4\ \mathbf{w}_5]; \mathbf{w}_4 = [0\,1\,0]^T; \mathbf{w}_5 = [0\,0\,1]^T; \\ W_2 \cap W_4 = \mathsf{span}\ \mathbf{w}_6;\ \ \mathbf{w}_6 = [0\,1\,0]^T; \\ W_3 \cap W_4 = \mathsf{span}\ \mathbf{w}_7;\ \ \mathbf{w}_7 = [0\,0\,1]^T. \end{array}$$

Condition $\mathbf{N}^T\mathbf{G}^T\mathbf{w}_j \geq 0$ fails only for $\mathbf{w}_3$, and, since also $\mathbf{N}^T\mathbf{G}^T\mathbf{w}_3 \leq 0$ fails, $\mathbf{w}_3$ must be discarded. Therefore, $CB(U_f)$ for this example is given by the vertices of $CH(\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4, \mathbf{w}_5, \mathbf{w}_6, \mathbf{w}_7\})$, i.e. $CB(U_f) = \{\mathbf{w}_1, \mathbf{w}_2\}$ (see fig.4–b). The object is accessible and detachable by means of motions in the first quadrant of the y, ϑ plane. However, since U_f has no interior points, the grasp is not S.A.D..

If the contact point $\mathbf{c}_4$ is removed, the fourth column of $\mathbf{GN}$ is deleted, and basis vectors for the two-by-two intersections of nullspaces are $\mathbf{w}_1, \mathbf{w}_2$, and $\mathbf{w}_3$. Since in this case $\mathbf{N}^T\mathbf{G}^T\mathbf{w}_3 \leq 0$, the sign of $\mathbf{w}_3$ is reversed. A convex basis of U_f in this case is given by $CB(U_f) = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ (see fig.4–c), and the grasp is S.A.D.

Example 3. For the example in fig.5–a, contacts are placed in $\mathbf{c}_1 = [-1\ 0]^T$; $\mathbf{c}_2 = [-1/2\ \ -\sqrt{3}/2]^T$; $\mathbf{c}_3 = [1/2\ \ -\sqrt{3}/2]^T$, and the associated directions are $\mathbf{n}_1 = [1\ 0]^T$; $\mathbf{n}_2 = [1/2\ \sqrt{3}/2]^T$; $\mathbf{n}_3 = [-1/2\ \sqrt{3}/2]^T$. The grasp constraint matrix is

$$\mathbf{GN} = \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & \sqrt{3}/2 & \sqrt{3}/2 \\ 0 & 0 & 0 \end{bmatrix},$$

and possesses a 1–dimensional nullspace $W = \mathsf{span}\ [0\ 0\ 1]^T$. The nullspaces of the individual contact constraints are

$$W_1 = \mathsf{span} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}; W_2 = \mathsf{span} \begin{bmatrix} 0 & \sqrt{3} \\ 0 & -1 \\ 1 & 0 \end{bmatrix}; W_3 = \mathsf{span} \begin{bmatrix} 0 & \sqrt{3} \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The intersections of these subspaces should be taken $(d-h-1=3-2=)$ 1 at a time, therefore the $\mathbf{w}_j$'s coincide with the basis vectors of W_i. A basis of W already appears on the first columns of such bases; let $\mathbf{w}_1 = [0\,0\,1]^T$; $\mathbf{w}_2 = [0\,1\,0]^T$; $\mathbf{w}_3 = [\sqrt{3}\ -1\,0]^T$; $\mathbf{w}_4 = [\sqrt{3}\,1\,0]^T$. Vector $\mathbf{w}_3$ fails condition $\mathbf{N}^T\mathbf{G}^T\mathbf{w}_j \geq 0$ as well as $\mathbf{N}^T\mathbf{G}^T\mathbf{w}_3 \leq 0$, hence $\mathbf{w}_3$ is discarded. Since the vertices of $CH(\{\mathbf{w}_2, \mathbf{w}_4\})$ are simply $\{\mathbf{w}_2, \mathbf{w}_4\}$, a convex basis of U_f is given by $CB(U_f) = \{\mathbf{w}_1, -\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$ (see fig.5–b). By application of proposition 6, the grasp is S.A.D.

Example 4. To illustrate a limit case in the application of algorithm 1, consider the example depicted in fig.6–a, where two contacts are applied at the same point $\mathbf{c}_1 = \mathbf{c}_2 = [-1\ 0]^T$, that inhibit motions along the direction $\mathbf{n}_1 = \mathbf{n}_2 = [1\ 0]^T$, and a third contact is placed in $\mathbf{c}_3 = [1\ 0]^T$ with $\mathbf{n}_3 = [-1\ 0]^T$. In this case the grasp constraint matrix is

$$\mathbf{GN} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

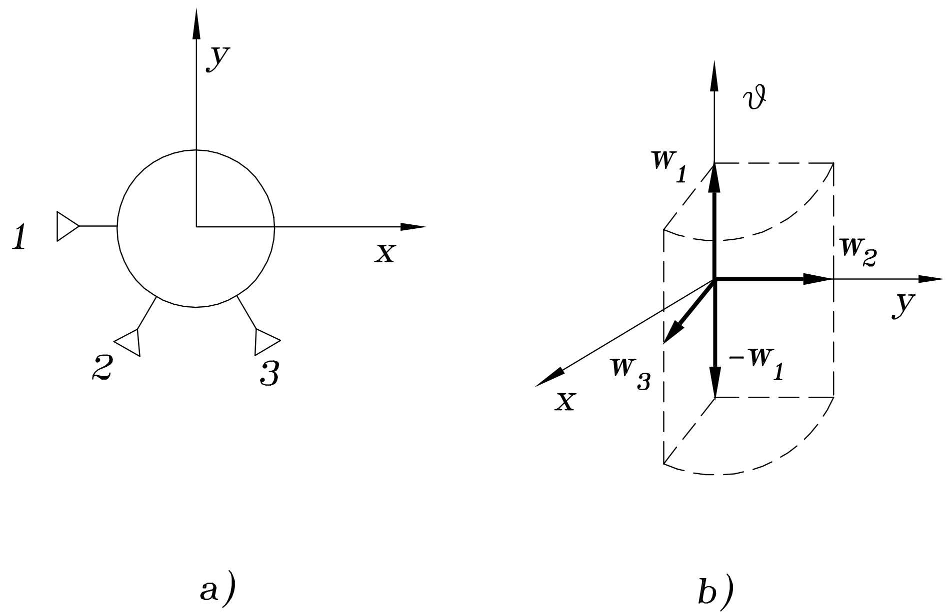


Figure 5: Example 3.

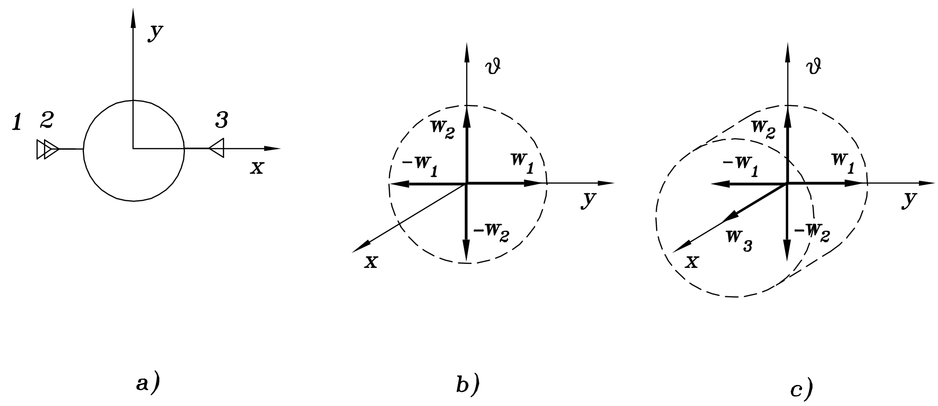


Figure 6: Example 4.

and there is a 2-dimensional nullspace $W = \mathsf{span}\,[\mathbf{w}_1, \mathbf{w}_2]$, $\mathbf{w}_1 = [0\ 1\ 0]^T$, $\mathbf{w}_2 = [0\ 0\ 1]^T$. Since in this example $d-h-1=0$, we take $\mathbf{w}_3 = \mathbf{G}^T\mathbf{n}_1 = [1\,0\,0]^T$, $\mathbf{w}_4 = \mathbf{G}^T\mathbf{n}_2 = [1\,0\,0]^T$, $\mathbf{w}_5 = \mathbf{G}^T\mathbf{n}_3 = [-1\ 0\ 0]^T$. This vectors are all discarded through the test $\mathbf{N}^T\mathbf{G}^T\mathbf{w}_j \geq 0$, hence we have $CB(U_f) = \{\mathbf{w}_1, \mathbf{w}_2, -\mathbf{w}_1, -\mathbf{w}_2\}$ (see fig.6–b) and, by proposition 6, the set of constraints is A.D. but not S.A.D..

If the contact in $\mathbf{c}_3$ is removed, and the third column of $\mathbf{GN}$ deleted, vectors $\mathbf{w}_3$ and $\mathbf{w}_4$ pass the test $\mathbf{N}^T\mathbf{G}^T\mathbf{w}_j \geq 0$. The only vertex of $CH(\{\mathbf{w}_3, \mathbf{w}_4\})$ is $\mathbf{w}_3$. Therefore, $CB(U_f) = \{\mathbf{w}_1, \mathbf{w}_2, -\mathbf{w}_1, -\mathbf{w}2, \mathbf{w}_3\}$ (see fig.6–c), and the grasp is S.A.D..

Example 5. If the seventh contact is removed from the set of constraints on the cube of fig.2, it is expected from Reuleaux's condition (1) that the maximal form-restrained set is at most 5-dimensional. The constraint matrix is given by

$$\mathbf{GN} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and is still full rank. Intersecting the individual constraint nullspaces, we obtain

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix};\ \mathbf{w}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix};\ \mathbf{w}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix};\ \mathbf{w}_4 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix};\ \mathbf{w}_5 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix};\ \mathbf{w}_6 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Condition $\mathbf{N}^T\mathbf{G}^T\mathbf{w}_j \geq 0$ holds for all $j = 1, \ldots, 6$ in this case. Moreover, all $\mathbf{w}_j$'s are linearly independent, therefore they form a minimal convex basis for the cone U_f, $CB(U_f) = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5, \mathbf{w}_6\}$. Since dim span $[CB(U_f)] = 6$, the set of constraints is S.A.D..

3 Force–Closure

The analysis of force–closure differs from that of form–closure because it takes into account the capability of the grasping mechanism to actively control some of the contact forces. The force and moment balance equations for an object subject to an external force $\mathbf{f}$ and moment $\mathbf{m}$, while grasped by a robotic mechanism by means of n contact forces $\mathbf{p}_i$ applied at $\mathbf{c}_i$, can be written in matrix notation as

$$\mathbf{w} = \mathbf{G}\mathbf{p}, \tag{7}$$

where $\mathbf{w} = (\mathbf{f}^T, \mathbf{m}^T)^T$ is the so-called external wrench, and $\mathbf{p} = (\mathbf{p}_1^T, \ldots, \mathbf{p}_n^T)^T$. The relationship between contact forces and the torques at the m joints of the robotic hand can be written as

$$\tau = \mathbf{J}^T\mathbf{p},$$

where $\mathbf{J}$ is equivalent to the jacobian matrix for conventional manipulators. A general solution of (7) can be written in the hypothesis that $\mathbf{w}$ is *resistible* (i.e., that rank $\mathbf{G} =$ rank $[\mathbf{G}\ \mathbf{w}]$) as

$$\mathbf{p} = \mathbf{G}^R\mathbf{w} + \mathbf{A}\mathbf{x}, \tag{8}$$

i.e., the sum of a particular solution of (7) ($\mathbf{G}^R$ is a right-inverse of $\mathbf{G}$), and a homogeneous solution. $\mathbf{A}$ is a matrix whose column form a basis of the nullspace of $\mathbf{G}$. The coefficient vector $\mathbf{x} \in \mathbb{R}^{h_0}$ parametrizes the homogeneous solution. Internal contact forces $\mathbf{p}_h = \mathbf{A}\mathbf{x}$ have no direct effect on the external wrench $\mathbf{w}$, but play an important role in the robustness of the equilibrium with respect to slippage induced by external disturbances, by allowing to "squeeze" the object in the grasp. However, as pointed out in the introduction by examples in fig.1–c, d, not always is it possible to apply arbitrary internal forces. It has been shown by the author [1993] that the subspace of homogeneous solutions of (7) can be subdivided in a subspace of active (controllable) internal forces and passive (non-controllable, or preload) internal forces. The latter are contact forces that cannot be influenced by joint torques, but may be present due to initial preloading of the grasp (as e.g. in a vise-like mechanical fixture) or to wedging effects. Let $\mathbf{E}$ be a matrix whose columns form a basis for the subspace of controllable internal forces, and assume, with no loss of generality, that preload forces are zero. The general solution to (7) can be rewritten as

$$\mathbf{p} = \mathbf{G}^R\mathbf{w} + \mathbf{E}\mathbf{y}, \tag{9}$$

where $\mathbf{y} \in \mathbb{R}^h$, and $h \leq h_0$. The controllability of internal forces for general grasping mechanisms has been considered by the author [1993], where a "virtual–spring" elastic model is assumed for contact forces that is embodied in a grasp stiffness matrix $\mathbf{K}$. It can be shown by both a dynamic and a quasi–static analysis that the subspace of internal forces controllable at equilibrium $\mathcal{F}_{hc}$ can be expressed in terms of the nullspace of the grasp matrix $\mathcal{N}(\mathbf{G})$ and of the range spaces of $\mathbf{KJ}$ and $\mathbf{KG}^T$ as

$$\mathcal{F}_{hc} = \mathcal{N}(\mathbf{G}) \cap \left(\mathcal{R}(\mathbf{KJ}) + \mathcal{R}(\mathbf{KG}^T)\right). \tag{10}$$

Algorithms for evaluating a basis of $\mathcal{F}_{hc}$ and building matrix $\mathbf{E}$ are discussed in the above reference.

In force–closure analysis one generally has to deal with frictional contacts. In general, contacts of the contact-point-with-friction, soft-finger, or very-soft-finger (complete-constraint) type ([Cutkosky, 1985]) can be assumed to be in effect. Accordingly, friction forces and torques will be subject to limitations due to Coulomb's law of friction or to its generalizations ([Goyal, 1989], [Howe, *et al.* 1988]) . However, in this paper we will only consider contacts of the first type, since the generalization poses no difficulties. For point-contact-with-friction, the constraint is described by Coulomb's inequality,

$$\sigma_{i,f}(\mathbf{p}_i) \;=\; \alpha_i\|\mathbf{p}_i\| - \mathbf{p}_i^T\mathbf{n}_i < 0, \tag{11}$$

representing a cone in the space of contact forces $\mathbf{p}_i$. A friction constraint is said to be "marginal" when $\mathbf{p}_i$ lies on the boundary of the friction cone, i.e. when $\sigma_{i,f}(\mathbf{p}_i) = 0$. By partitioning (9) as

$$\begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_n \end{bmatrix} = \begin{bmatrix} \mathbf{P}_1 \\ \vdots \\ \mathbf{P}_n \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{M}_1 \\ \vdots \\ \mathbf{M}_n \end{bmatrix} \mathbf{y},$$

we have

$$\mathbf{p}_i(\mathbf{w},\mathbf{y}) = \mathbf{P}_i\ \mathbf{w} + \mathbf{M}_i\ \mathbf{y}. \tag{12}$$

Substituting (12) in (11), we obtain the expression of friction constraints $\sigma_{i,f}(\mathbf{w},\mathbf{y}) < 0$. If the term "grasp" is used to identify a grasping mechanism, a set of contact points and a set of friction constraints, we introduce the following

Definition 4 *A grasp is defined* Force–Closure *if, for any external wrench* $\mathbf{w}$ *acting on the object, there exists a vector* $\mathbf{y}$ *such that all friction constraints are fulfilled.*

This definition can be generalized to partially restraining sets of constraints:

Definition 5 *A grasp is* partially force–closure *with respect to wrenches in a subspace* $W \subset \mathbb{R}^d$ *if, for any* $\mathbf{w} \in W$ *acting on the object, there exist a vector* $\mathbf{y}$ *such that all friction constraints are fulfilled.*

In the analysis of force–closure, it is useful to consider the so-called *prehensility* of the grasp [Murray and Sastry, 1990].

Definition 6 *A grasp is said to be prehensile if a vector* $\mathbf{y}$ *exists such that* $\sigma_{i,f}(0, \mathbf{y}) < 0, \forall i$.

In other words, prehensility expresses the fact that controllable, purely internal forces exist that comply with the friction constraints. For a grasp with grasp matrix $\mathbf{G}$, we can now prove the following

Proposition 7 *A grasp is partially force–closure with respect to* $W = \mathsf{range}\ (\mathbf{G})$ *if and only if it is prehensile.*

Proof: The "only if" part is trivial. Assume that, for $\mathbf{w} = 0$ and $\mathbf{y} = \bar{\mathbf{y}}$, the i-th friction constraint evaluates to

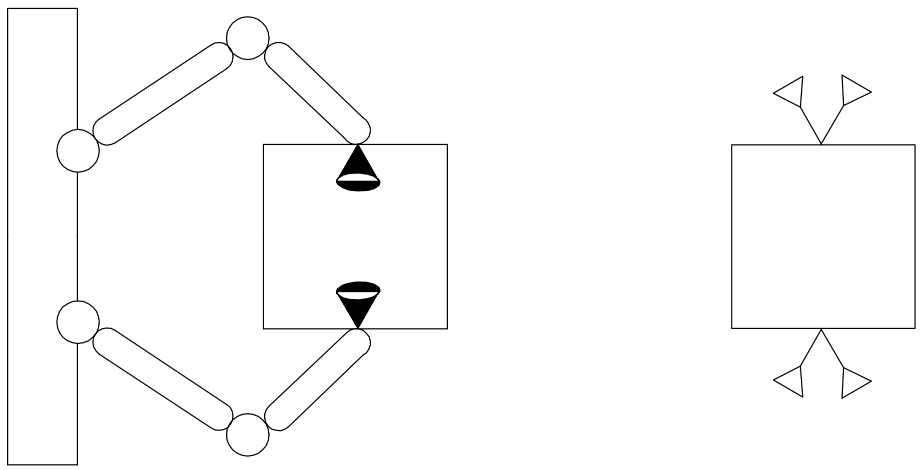
$$\sigma_{i,f}(0, \bar{\mathbf{y}}) \ = \ \alpha_i \|\mathbf{M}_i \bar{\mathbf{y}}\| - \bar{\mathbf{y}}^T \mathbf{M}_i^T \mathbf{n}_i = -\delta_i < 0.$$

For an arbitrary $\mathbf{w} \in W$, define $k = \max_i \frac{\alpha_i \|\mathbf{P}_i \mathbf{w}\|}{\delta_i}$, and set the internal forces to $\mathbf{p} = k\mathbf{E}\bar{\mathbf{y}}$. It can be easily verified that $\sigma_{i,f}(\mathbf{w}, k\bar{\mathbf{y}}) < 0$, which proves the "if" part. □

Corollary 1 *If a grasp is prehensile and* $\mathbf{G}$ *is full row rank, the grasp is force–closure.*

Checking wheter a given grasp is prehensile is an important issue in both planning and control of grasping and manipulation operations. It is important to note that, for 2D grasps with non-defective end-effectors (i.e., $h = h_0$), a force–closure problem can be reformulated in terms of form–closure, for which efficient linear programming techniques are available. In fact, it is sufficient to replace each 2D frictional contact with two frictionless constraints, applied at the same point, preventing motions along the edges of the friction sector (see fig.7). In this sense, checking prehensility (and hence force–closure) in the plane is equivalent to checking form–closure for a suitably modified set of constraints. Examples of force–closure tests derived from this observation are those reported by Ferrari and Canny [1992] and Chen and Burdick [1993].

This approach cannot be used if internal forces are not all controllable, neither it can be extended to 3D grasps, since no finite set of edge vectors can positively span a 3D friction cone. A possible approximate technique for non-defective, 3D grasps consists in replacing friction cones with pyramids (see e.g. Kerr and Roth, [1986]). The method however trades accuracy for complexity, and a very large number of constraints in the equivalent form–closure model must be expected for reasonable approximations of general n-contacts grasps.



a) b)

Figure 7: A 2D force–closure problem and its equivalent form–closure problem

To our knowledge, the only method so far presented that investigates exactly the force–closure property of 3–D, n-contacts grasps, was proposed by Nakamura *et al.* [1989]. Their method can easily be extended to defective end-effectors, provided that the basis of controllable internal forces $\mathbf{E}$ is used in place of the basis of the nullspace of $\mathbf{G}$. The algorithm consists of the solution of 12 nonlinear optimization problems with unilateral constraints. Considering that in the evaluation of different possible grasps the force–closure test can be easily needed tens of times, the usefulness of a faster algorithm is apparent.

In order to achieve such goal, let us define an auxiliary constraint on the minimum value $f_{i,min} > 0$ of normal forces as

$$\sigma_{i,m}(\mathbf{p}_i(\mathbf{w}, \mathbf{y})) = f_{i,min} - \mathbf{p}_i^T \mathbf{n}_i < 0. \tag{13}$$

Since the set of homogeneous solutions that satisfy friction constraints is a cone (it consists of the cartesian product of the individual friction cones), this auxiliary constraint does not influence the prehensility property, and is only introduced for further convenience. In fact it can be easily proven the following

Lemma 1 *If a grasp is prehensile, then for all* $\mathbf{y}$ *such that* $\sigma_{i,f}(0, \mathbf{y}) < 0$ *and for all* $\zeta > 0$, *there exists* h *such that* $\sigma_{i,j}(0, h\mathbf{y}) < -\zeta, \ \forall i, j$.

Note that constraints (11) and (13) on the i-th contact force can be written in the same form

$$\sigma_{i,j}(\mathbf{y}) \ = \ \alpha_{i,j} \ \|\mathbf{p}_i\| + \ \beta_{i,j} \ \mathbf{p}_i^T \mathbf{n}_i + \ \gamma_{i,j} \ < \ 0, \tag{14}$$

where $\alpha_{i,f} = \alpha_i$, $\beta_{i,f} = -1$, and $\gamma_{i,f} = 0$ for friction constraints; $\alpha_{i,m} = 0$, $\beta_{i,m} = -1$, and $\gamma_{i,m} = f_{i,min}$ for minimum force constraints. Let $\Omega^{\kappa}_{i,j} \subset \mathbb{R}^h$ indicate the set of grasp variables that satisfy constraints (14) of corresponding indices with a (small, positive) margin κ,

$$\Omega^{\kappa}_{i,j} \;:=\; \{\mathbf{y} \mid \sigma_{i,j}(\mathbf{w},\mathbf{y}) < -\kappa\}.$$

For the i-th contact and the j-th constraint, consider the functions

$$V_{i,j}(\mathbf{w},\mathbf{y}) = \begin{cases} (2\;\sigma^2_{i,j}(\mathbf{w},\mathbf{y}))^{-1} & \mathbf{y} \;\in\; \Omega^{\kappa}_{i,j} \\ a\;\sigma^2_{i,j}(\mathbf{w},\mathbf{y}) + b\;\sigma_{i,j}(\mathbf{w},\mathbf{y}) + c & \mathbf{y} \;\notin\; \Omega^{\kappa}_{i,j} \end{cases}, \tag{15}$$

and associate to the grasp a function $V(\mathbf{w},\mathbf{y})$ defined as the summation of such terms:

$$V(\mathbf{w},\mathbf{y}) = \sum_{i=1}^{n} \sum_{j=f,m} V_{i,j}(\mathbf{w},\mathbf{y}), \tag{16}$$

Lemma 2 *(Necessary and sufficient condition for force–closure.) A grasp is force–closure if and only if, for all $\xi > 0$, there exists $\mathbf{y}$ such that $V(0,\mathbf{y}) < \xi$.*

This follows directly from lemma 1, and accounts for the introduction of the function V. An algorithm for checking the force–closure property can be based on the search for $\mathbf{y}$ that minimize the associate function V. This can be efficiently implemented exploiting the simple structure of V. In fact, the gradient of V with respect to $\mathbf{y}$ is the summation over i and j of the terms

$$\frac{\partial V_{i,j}}{\partial y} = \begin{cases} -\sigma^{-3}_{i,j}\;\frac{\partial\;\sigma_{i,j}}{\partial y}, & \mathbf{y} \in \Omega^{\kappa}_{i,j} \\ (2a\;\sigma_{i,j} + b)\;\frac{\partial\;\sigma_{i,j}}{\partial y} & \mathbf{y} \notin \Omega^{\kappa}_{i,j} \end{cases}, \tag{17}$$

where

$$\frac{\partial\;\sigma_{i,j}}{\partial y} = \alpha_{i,j}\;\mathbf{M}_i^T\vec{\mathbf{p}}_i + \;\beta_{i,j}\;\mathbf{M}_i^T\mathbf{n}_i, \tag{18}$$

and $\vec{\mathbf{p}} = \mathbf{p}/\|\mathbf{p}\|$. The hessian of V is the summation of the terms

$$\frac{\partial^2 V_{i,j}}{\partial y^2} = \begin{cases} -\sigma^{-3}_{i,j}\;\frac{\partial^2\;\sigma_{i,j}}{\partial y^2} + \;3\sigma^{-4}_{i,j}\;\frac{\partial\;\sigma_{i,j}}{\partial y}\;\frac{\partial\;\sigma^T_{i,j}}{\partial y} & \mathbf{y} \in \Omega^{\kappa}_{i,j} \\ (2a\sigma_{i,j} + b)\;\frac{\partial^2\;\sigma_{i,j}}{\partial y^2} + 2a\frac{\partial\;\sigma_{i,j}}{\partial y}\;\frac{\partial\;\sigma^T_{i,j}}{\partial y} & \mathbf{y} \notin \Omega^{\kappa}_{i,j} \end{cases}, \tag{19}$$

where

$$\frac{\partial^2\;\sigma_{i,j}}{\partial y^2} \;=\; \alpha_{i,j}\;\frac{\mathbf{M}_i^T\left(\mathbf{I}\;-\;\vec{\mathbf{p}}_i\vec{\mathbf{p}}_i^T\right)\mathbf{M}_i}{\|\mathbf{p}_i\|}.$$

Imposing twice continuous differentiability of $V_{i,j}(\mathbf{w},\mathbf{y})$ on the boundaries of $\Omega^{\kappa}_{i,j}$ provides conditions on a, b, and c.

Lemma 3 *The function $V(\mathbf{w},\mathbf{y})$ defined in (16) with $a = \frac{3}{2\kappa^4}$, $b = \frac{4}{\kappa^3}$, and $c = \frac{3}{\kappa^2}$, is strictly convex with respect to $\mathbf{y} \in \mathbb{R}^h$, for any $\mathbf{w} \in \mathbb{R}^d$.*

The proof follows from observing that the discontinous terms in (14), (17), and (19) can be regarded as the limits of sequences of functions continuously differentiable over $\mathbb{R}^h$. Hence, the positive definiteness of the hessian of V is a necessary and sufficient condition for its convexity. Being (19) the summation of matrices which can be trivially shown to be s.p.d., it will suffice to show that the intersection of the nullspaces of each addend is zero. Assume that there exists a vector $\mathbf{x} \in \mathbb{R}^h$ such that, for every i, j, $\mathbf{x}^T \frac{\partial^2 \sigma_{i,j}}{\partial y^2} \mathbf{x} = 0$, and $\mathbf{x}^T \frac{\partial \sigma_{i,j}}{\partial y} \frac{\partial \sigma_{i,j}^T}{\partial y} \mathbf{x} = 0$. Explicitly, for friction constraints, these relations imply

$$\left\{ \begin{array}{l} \mathbf{M}_i \mathbf{x} \;\; \text{parallel to} \;\; \vec{\mathbf{p}}_i \\ (\alpha_i \vec{\mathbf{p}}_i - \mathbf{n}_i)^T \mathbf{M}_i \mathbf{x} = 0 \end{array} \right. , \tag{20}$$

while, for minimum constraints, we have

$$\mathbf{n}_i^T \mathbf{M}_i \mathbf{x} = 0. \tag{21}$$

Conditions (20) and (21) together imply that $\mathbf{M}_i \mathbf{x}$ should be simultaneously parallel and normal to $\vec{\mathbf{p}}_i$. The only solution is $\mathbf{M}_i \mathbf{x} = 0$. Since this must hold for every i, by juxtaposing all such relationship we have the condition $\mathbf{E}\mathbf{x} = 0$. Being the columns of $\mathbf{E}$ independent (they form a basis of the subspace of homogeneous solutions), it follows $\mathbf{x}^T \frac{\partial^2 V}{\partial y^2} \mathbf{x} > 0, \forall \mathbf{x} \neq 0$. □

Based on the definitions and lemmata above, we are now in a position to state the technique for checking the force–closure property. This is based on the study of the equilibria of an ordinary differential equation which is associated with the grasp and is defined as

$$\dot{\mathbf{y}}(t) = -\zeta \left. \frac{\partial^2 V}{\partial y^2}^{-1} \right|_{0,\mathbf{y}} \left. \frac{\partial V}{\partial y} \right|_{0,\mathbf{y}}, \tag{22}$$

with $\zeta > 0$.

Theorem 1 *A necessary condition for the prehensility of a grasp is that the dynamics of the associate system (22) diverge. If* $\inf_t V(0, \mathbf{y}(t)) = 0$ *along the trajectories of the associate system, the condition is also sufficient.*

Proof: Since V has been shown to be strictly convex, the dynamics (22) either have an unique equilibrium point $\hat{\mathbf{y}}$, or diverge. In the first case (which, according to lemma 2, corresponds to a lack of prehensility), $\hat{\mathbf{y}}$ is globally asymptotically attractive for (22). In fact, introducing $\mathbf{e}(t) = \mathbf{y}(t) - \hat{\mathbf{y}}$, the p.d. Lyapunov candidate $V(0, \mathbf{e})$ obtained from (16), i.e.

$$\dot{V} = \frac{\partial V}{\partial e}^T \dot{\mathbf{e}} = -\zeta \left. \frac{\partial V}{\partial y}^T \right|_{0,\mathbf{y}} \left. \frac{\partial^2 V}{\partial y^2}^{-1} \right|_{0,\mathbf{y}} \left. \frac{\partial V}{\partial y} \right|_{0,\mathbf{y}} \tag{23}$$

is clearly negative definite. On the other hand, if (22) diverges, then $V(\dot{\mathbf{y}}(t)) < 0$ implies $\lim_{t \to \infty} V(0, \mathbf{y}(t)) = l \geq 0$. The case $l > 0$ corresponds to some (say $m \geq 1$) constraints $\sigma_{i,j}(0, \mathbf{y}(t))$ converging to a constant value, while the remaining $2n - m$ diverge to $-\infty$ (no $\sigma_{i,j}(0, \mathbf{y}(t))$ can diverge to $+\infty$ because $\dot{V}(0, \mathbf{y}(t)) < 0$). Note that only friction constraints are among those converging, because, should any minimum constraint

$\sigma_{i,m}(0, \mathbf{y}(t))$ converge, the corresponding friction constraint $\sigma_{i,f}(0, \mathbf{y}(t))$ would diverge to $+\infty$. Hence for m of the friction constraints it holds

$$\lim_{\|\mathbf{y}\| \to \infty} \sigma_{i,f}(0, \mathbf{y}) = \lim_{k \to \infty} \sigma_{i,f}(0, k\vec{\mathbf{y}}) = \lim_{k \to \infty} k \left(\alpha_i \|\mathbf{M}_i \vec{\mathbf{y}}\| - \mathbf{n}_i^T \mathbf{M}_i \vec{\mathbf{y}} \right) = 0$$

implying $\alpha_i \|\mathbf{M}_i \vec{\mathbf{y}}\| = \mathbf{n}_i^T \mathbf{M}_i \vec{\mathbf{y}}$, i.e., the m friction constraints are marginal, $\sigma_{i,f}(0, \mathbf{y}(t)) \equiv 0$. Therefore, according to (15), $\lim_{t \to \infty} V(0, \mathbf{y}(t)) = l = mc = \frac{3m}{\kappa^2}$. This case corresponds to the very particular case that both the contact forces generated by the external load and the internal forces lie on the boundaries of the friction cones. As soon as the external wrench is slightly modified, some friction constraints become violated, whatever internal force is applied. Therefore such grasps are not force–closure.
If $\inf_t V(0, \mathbf{y}(t)) = l = 0$, the grasp is prehensile by lemma 2. □

Algorithm 2 (Force–Closure Test) Set up the associate dynamic system (22) as a difference equation, and integrate it numerically starting from arbitrary initial conditions. At the k-th step, check whether $\mathbf{y}_k$ satisfies all friction constraints: if so, conclude for prehensility and stop the algorithm. Force–closure is checked using Corollary 1. If otherwise $\mathbf{y}_k$ converges to an equilibrium $\hat{\mathbf{y}}$ (then necessarily $\sigma_{i,f} > 0$ for some i), exclude prehensility. The (unlikely) case of convergence to a finite value $l > 0$ leads the algorithm to a stall ($\left.\frac{\partial V}{\partial y}\right|_{0,\mathbf{y}_k} \simeq 0$ for large k), and is easily recognized ($V(0, \mathbf{y}_i) \geq \frac{3}{\kappa^2}$ for large i).

Remark I The dynamics of system (22) can be made arbitrarily fast by increasing ζ. However, in its difference equation realization, the global asymptotic convergence of the algorithm can be proven only for values of ζ smaller than a limit value. Such limitations on ζ only pose minor problems in practical applications of the algorithm.

4 Quantitative test

Closure tests discussed so far in this paper only provided a qualitative (true/false) answer. However, several researchers have pointed out that associating a quality index to a grasp is desirable if a choice is to be made among different grasping configurations (optimal grasp planning). Quantitative closure tests have been proposed for form–closure (Kirkpatrick *et al.* [1989]; Markenscoff and Papadimitriou [1989]; Trinkle [1992]) and for force–closure (Ferrari and Canny [1992]). As noted already, the latter method is based on the reduction of force–closure to an equivalent form–closure problem, and is close in spirit to the former group. A quantitative exact force–closure test, albeit not explicitly stated as such, can be derived from the work of Nakamura *et al.* [1989]. A force–closure quality index can also be obtained from the approach presented in section 3 of this paper, by simply including an upper bound of the contact forces that can be exerted by the end-effector on the object. The physical motivation for considering such bound is manifold, and includes possible limitations on actuator torques or power expenditure, fragility of the object, and the fact that hard squeezing may make gripping less stable (as it happens with a soap bar, for

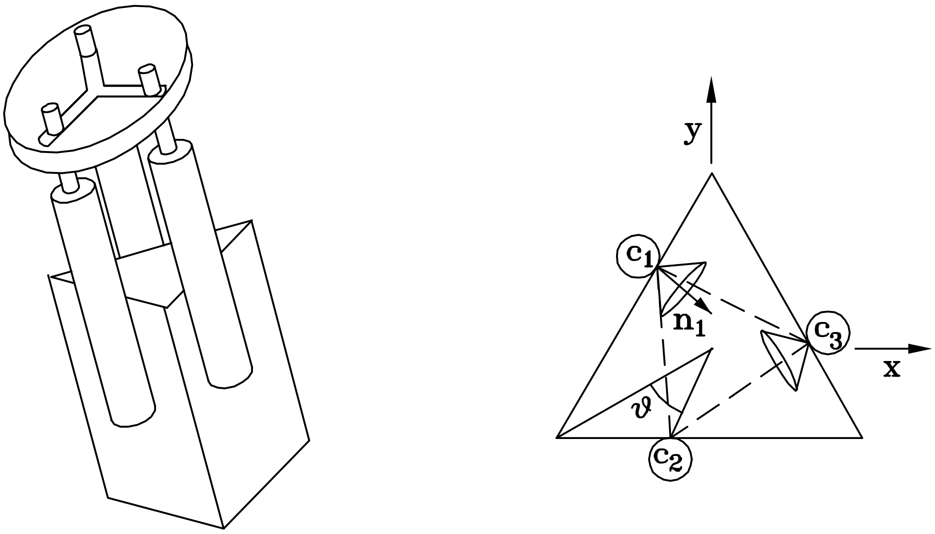


Figure 8: A three–fingered hand grasping a triangular cylinder.

instance). Mathematically, this bound can be written in terms of the maximum intensity of the i-th contact force, $f_{i,max} > 0$, as

$$\|\mathbf{p}_i\| \leq f_{i,max},$$

and hence be cast in the form (14) by setting $\alpha_{i,M} = 1$, $\beta_{i,M} = 0$, and $\gamma_{i,M} = -f_{i,max}$. To this constraints, s.p.d. functions $V_{i,M}(\mathbf{w}, \mathbf{y})$ can be associated as in (16), and a new global function $\bar{V}(\mathbf{w}, \mathbf{y})$ is defined by extending the summation in (16) to index M. This new $\bar{V}(\mathbf{w}, \mathbf{y})$ is still strictly convex, but it is now radially unbounded. Hence, any trajectory of the dynamic system

$$\dot{\mathbf{y}}(t) = -\zeta \left. \frac{\partial^2 \bar{V}}{\partial y^2}^{-1} \right|_{0,\mathbf{y}} \left. \frac{\partial \bar{V}}{\partial y} \right|_{0,\mathbf{y}}, \tag{24}$$

will converge (with a second–order rate) to a unique, globally attractive equilibrium point $\bar{\mathbf{y}}$. The inverse of $\bar{V}(\mathbf{w}, \bar{\mathbf{y}})$ is therefore a well-defined, exact force–closure quality index that reflects the "distance" of the grasp from violating contact constraints. Its evaluation implies numerical simulation of the dynamics (24), that is only slightly more time–consuming than running the qualitative force–closure test of algorithm 2.

Example 6. Consider the grasp of the object depicted in fig.8 (a cylinder with equilateral triangular cross section), by means of a three–fingered star–shaped gripper. The fingers are placed at the vertices of an equilateral triangle, rotated of an angle θ with respect to the object. This type of grippers, often used in industry, usually have one degree–of–freedom only, and the fingers are constrained to move along straight lines departing form the gripper centre. The range of θ giving force–closure grasps is investigated, and the optimal θ is sought. Assuming unitary length for the sides of the object cross-section, contact points are placed in

$$\mathbf{c}_1 = \begin{bmatrix} -a\sin(\theta) \\ a\cos(\theta) \\ 0 \end{bmatrix}; \mathbf{c}_2 = \begin{bmatrix} -a\cos(30+\theta) \\ -a\sin(30+\theta) \\ 0 \end{bmatrix}; \mathbf{c}_3 = \begin{bmatrix} a\cos(30-\theta) \\ -a\sin(30-\theta) \\ 0 \end{bmatrix},$$

where $a = \frac{1}{2\sqrt{3}\sin(30+\theta)}$. Contact normals are $\mathbf{n}_1 = [\sqrt{3}/2 \;\; -1/2 \; 0]^T$; $\mathbf{n}_2 = [0\;1\;0]^T$; $\mathbf{n}_3 = [-\sqrt{3}/2 \;\; -1/2 \; 0]^T$. The Jacobian of this type of gripper is

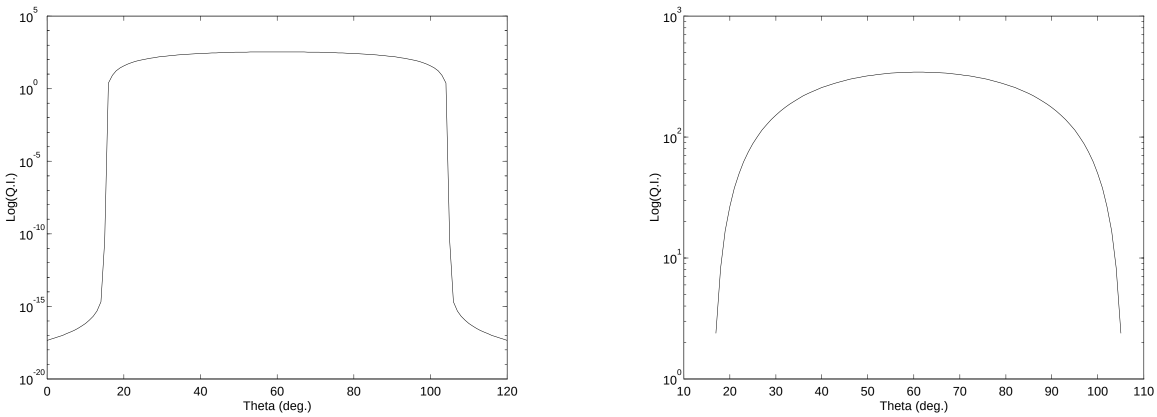
$$\mathbf{J} = \begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \\ 0 \\ \cos(30+\theta) \\ \sin(30+\theta) \\ 0 \\ -\cos(30-\theta) \\ \sin(30-\theta) \\ 0 \end{bmatrix}.$$

Although the nullspace of the grasp matrix is three-dimensional, the controllable internal force subspace is only one-dimensional. In this particularly simple case, if isotropic elastic properties are assumed for all bodies in contact, it results $\mathbf{E}(\theta) = \mathbf{J}(\theta)$. We assume a coefficient of friction $\alpha_i = \sqrt{2}/2$, $i = 1, \ldots, 3$ (corresponding to friction cones with 45 deg. half–angle), and choose $\mathbf{f}_{min} = 0.1$, $\mathbf{f}_{max} = 100$, $\kappa = 10^{-5}$. By applying the quantitative test algorithm of section 4 to a set of grasp configurations with 0 deg.$< \theta <$120 deg. in steps of 1 deg., the plots of the quality index $\bar{V}^{-1}$ vs. θ reported in fig.9 are obtained. The equilibrium point $\hat{y}$ reached by the dynamics associated with each grasp configuration are chosen as initial conditions for the next configuration, so that convergence is very rapid. In fig.9–a steep variations of the cost function in $\theta = 15$deg. and $\theta = 105$deg. can be noted, corresponding to a distinction between unstable grasps (0 deg.$< \theta <$15 deg. and 105 deg.$< \theta <$120 deg.) and force–closure grasps (15 deg.$< \theta <$105 deg.). Also, from the zoom of the central part of the plot reported in fig.9–b, it appears that an optimal grasp is obtained for $\theta = 60$deg., i.e. for fingers placed in the middle of the object faces.

5 Conclusions

This paper reports on a systematic investigation on the closure properties of grasping. Definitions of form–closure and force–closure properties are chosen, among several existing in the literature, as those that appear to overlap least and to provide best insight. The concept of partial closure has been studied in more depth than it had been previously, and an algorithm for describing the geometry of partial form–closure grasps has been presented. Probably, however, the main contribution of this paper consists of the efficient algorithm for testing force–closure, that exploits the analogy with the behaviour of a purposefully designed dynamic system.

Results reported in this paper leave a number of open problems in the analysis and synthesis of closure grasps. In particular, as already noted, these results are only valid



a) b)

Figure 9: Force–closure quality index plotted vs. gripper angle for the example of fig.8

to the first–order, and the effects of rolling of finite–curvature fingers on the object are not taken into account. The role played by higher–order properties of the surfaces in contact has been studied by Montana [1992] and applied to closure analysis by Rimon and Burdick [1993]. Results presented so far are rather complex from a computational point of view, and further analysis is needed to provide algorithms suitable for implementation in a manipulation planning system. Due to the comparative simplicity of first–order analysis, and to the fact that it provides conservative answers to closure tests (first–order closure implies closure), the role of first–order tests is probably to hold a central place in grasp planning. Another direction left unexplored by this paper is how information about the mobility of the object under partial closure grasps can be exploited for planning dexterous manipulations. Finally, an open question lies behind the usage of the stiffness matrix **K** made in the algorithm for calculating the subspace (10) of controllable internal forces. Although methods for computing **K** have been provided by Cutkosky and Kao [1989], stiffness data are not usually known. Suitable identification techniques should be developed for estimating such data if sensors are available to measure forces at the fingers. Fortunately, experimental results seem to indicate that the geometry of the space of contact forces is not too sensitive to errors in **K**, but a quantitative robustness analysis on this point is lacking.

Acknowledgements

The research reported in this paper has been partially supported by the C.N.R. - Progetto Finalizzato Robotica under contract 93.01079.PF67 and 93.01047.PF67.

References

Asada, H., and By, A.B. 1985. Kinematic Analysis of Workpart Fixturing for Flexible Assembly with Automatically Reconfigurable Fixtures. *IEEE Trans. on Robotics and Automation*, vol. 1, no. 2, pp. 86–94.

Baker, B.S., Fortune, S., and Grosse, E. 1985. Stable Prehension with a Multifingered Hand. *Proc. IEEE Int. Conf. on Robotics and Automation*, pp. 570–575.

Bicchi, A. 1993. Analysis of Force Distribution in Whole-Mechanism Manipulation Systems. *Proc. IEEE Int. Conf. on Robotics and Automation*, pp. 228–233.

Brock, D.L. 1988. Enhancing the dexterity of a Robot hand Using Controlled Slip. *Proc. IEEE Int. Conf. on Robotics and Automation*, pp. 249–251.

Brost, R.C. 1988. Automatic Grasp Planning in the Presence of Uncertainty. *Int. J. of Robotics Res.*, vol. 7, no. 1, pp. 3–17.

Chen, I-M., and Burdick, J.W. 1993. A Qualitative Test for N-Finger Force–Closure Grasps on Planar Objects with Applications to Manipulation and Finger Gaits. *Proc. IEEE Int. Conf. on Robotics and Automation*, pp. 814–820.

Cheng, F-T, and Orin, D.E. 1990. Efficient Algorithm for Optimal Force Distribution — The Compact–Dual LP method. *IEEE Trans. on Robotics and Automation*, vol. 6, no. 2, pp. 178–187.

Cole, A.A., Hsu, P., and Sastry, S.S. 1992. Dynamic Control of Sliding by Robot Hands for Regrasping. *IEEE Trans. on Robotics and Automation*, vol. 8, no. 1, pp. 42–52.

Cutkosky, M.R. 1985. *Robotic Grasping and Fine Manipulation.* Boston: Kluwer Academic.

Cutkosky, M.R., and Kao, I. 1989. Computing and Controlling the Compliance of a Robotic Hand. *IEEE Trans. on Robotics and Automation*, vol. 5, no. 2, pp. 151–165.

Fearing, R. 1986. Simplified Grasping and Manipulation with Dextrous Robot Hands. *IEEE J. Robotics and Automation*, vol. 2, no. 4, pp. 188–195.

Ferrari, C., and Canny, J. 1992. Planning Optimal Grasps. *Proc. IEEE Int. Conf. on Robotics and Automation*, pp. 2290–2295.

Gale, D. 1960. *The Theory of Linear Economic Models.* New York: McGraw Hill.

Goldman, A.J., and Tucker, A.W. 1956. Polyhedral Convex Cones. *Linear Inequalities and Related Systems*, eds. Kuhn, H.W., and Tucker, A.W. Annals of Mathematics Studies 38, Princeton University Press.

Goyal, S. 1989. Planar Sliding of a Rigid Body with Dry Friction: Limit Surfaces and dynamics of motion. Ph.D. Thesis, Cornell University.

Hirai, S., and Asada, H. 1993. Kinematics and Statics of Manipulation Using the Theory of Polyhedral Convex Cones. *Int. J. of Robotics Res.*, vol. 12, no. 5, pp. 434–447.

Howe, R.D., Kao, I., and Cutkosky, M.R. 1988. The Sliding of Robot Fingers Under Combined Torsion and Shear Loading. *Proc. IEEE Int. Conf. on Robotics and Automation*, pp. 103–105.

Hunt, K.H., Samuel, A.E., and McAree, P.R. 1991. Special configurations of Multi–finger Multi–freedom Grippers — A Kinematic Study. *Int. J. of Robotics Res.*, vol. 10, no. 2, pp. 123–134.

Kerr, J., and Roth, B. 1986. Analysis of Multifingered Hands. *Int. J. of Robotics Research*, vol. 4, no. 4, pp. 3–17.

Kirkpatrick, D., Kosaraju, S.R., Mishra, B., and Yap, C–K. 1989. Quantitative Steinitz's Theorem with Applications to Multifingered Grasping. TR460. New York: Dept. of Computer Science, Courant Inst. of Mathematical Sciences, New York University.

Lakshminarayana, K. 1978. Mechanics of Form Closure. *ASME paper 78-DET-32*, pp. 2–8.

Markenscoff, X., and Papadimitriou, C.H. 1989. Optimum Grip of a Polygon. *Int. J. of Robotics Res.*, vol. 8, no. 2, pp. 17–29.

Markenscoff, X., Ni, L., and Papadimitriou, C.H. 1990. The Geometry of Grasping. *Int. J. of Robotics Res.*, vol. 9, no. 1, pp. 61–74.

Mishra, B., Schwartz, J.T., and Sharir, M. 1987. On the Existence and Synthesis of Multifinger Positive Grips. *Algorithmica*, vol. 2, no. 4, pp. 541–558.

Montana, D.J. 1992. Contact Stability for Two–Fingered Grasps. *IEEE Trans. on Robotics and Automation*, vol. 8, no. 4, pp. 421–430.

Murray, R.M., Li, Z., and Sastry, S.S. 1994. *A Mathematical Introduction to Robotic Manipulation.* Boca Raton: CRC Press.

Nakamura, Y., Nagai, K., and Yoshikawa, T. 1989. Dynamics and Stability in Coordination of Multiple Robotic Systems. *Int. J. of Robotic Res.*, vol. 8, no. 2, pp. 44–61.

Nguyen, V.D. 1986. The synthesis of Stable Force–Closure Grasps. TR 905. Cambridge (MA): MIT A.I. Lab.

Nguyen, V.D. 1988. Constructing Force–Closure Grasps. *Int. J. of Robotics Res.*, vol. 7, no. 3, pp. 3–16.

Ohwovoriole, M.S. 1980. An Extension of Screw Theory and its Application to the Automation of Industrial Assemblies. Ph.D. Thesis, Stanford University, Dept. of Mechanical Engineering.

Park, Y.C., and Starr, G.P. 1992. Grasp Synthesis of Polygonal Objects Using a Three-Fingered Robot Hand. *Int. J. of Robotics Res.*, vol. 11, no. 3, pp. 163–184.

Pollard, N.S., and Lozano–Perez, T. 1990. Grasp Stability and Feasibility for an Arm with an Articulated Hand. *Proc. IEEE Int. Conf. on Robotics and Automation*, pp. 581–585.

Ponce, J., Stam, D., and Faverjon, B. 1993. On Computing Two-Finger Force-Closure Grasps of Curved 2D Objects. *Int. J. of Robotics Res.*, vol. 12, no. 3, pp. 263–273.

Rimon, E., and Burdick, J.W. 1993. Towards Planning with Force Constraints: On the Mobility of Bodies in Contact. *Proc. IEEE Int. Conf. on Robotics and Automation*, pp. 994–1000.

Reuleaux, F. 1875. Kinematics of Machinery. New York: Dover, 1963. First published in German, 1875.

Salisbury, J.K. 1982. Kinematic and Force Analysis of Articulated Hands. Rep. STAN-CS-82-91, Stanford, CA: Stanford University.

Selig, A.J., and Rooney, P.K. 1989. Reuleaux pairs and surfaces that cannot be gripped. *Int. J. of Robotics Res.*, vol. 8, no. 5, pp. 79–87.

Somov, P. 1900. Über Gebiete von Schraubengeschwindigkeiten eines starren Körpers bei verschiedener Zahl von Stützflächen. *Zeitschrift für Mathematik und Physik*, vol. 45.

Tournassoud, P., Lozano–Perez, T., and Mazer, E. 1987. Regrasping. *Proc. IEEE Int. Conf. on Robotics and Automation*, pp. 1924–1928.

Trinkle, J.C., Abel, J.M., and Paul, R.P. 1987. An Investigation of Frictionless Enveloping Grasping in the Plane. *Int. J. of Robotics Res.*, vol. 7, no. 3, pp. 33-51.

Trinkle, J.C. 1992. On the Stability and Instantaneous Velocity of Grasped Frictionless Objects. *IEEE Trans. on Robotics and Automation*, vol. 8, no. 5, pp. 560–572.

Waldron, K.J., Raghavan, M., and Roth, B. 1989. Kinematics of a hybrid series–parallel manipulation system. *J. Dyn. Sys. Meas. Control*, vol. 111, no. 2, pp. 211–221.