Optimal planning for coordinated vehicles with bounded curvature

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In this paper we consider the problem of planning motions of a system of multiple vehicles moving in a plane. Each vehicle is modelled as a kinematic system with velocity constraints and curvature bounds. Vehicles can not get closer to each other than a predefined safety distance. For such system of multiple vehicles, we consider the problem of planning optimal paths in the absence of obstacles. The case when a constant distance between vehicles is enforced (such as when co-operative manipulation of objects is performed by the vehicle team) is also considered.

1 Introduction

In this paper we consider the problem of planning motions of a system of multiple vehicles moving in a plane. Motion of each vehicle are subject to some constraints: the velocity of the center of the vehicle is parallel to an axis fixed on the vehicle; the velocity is constant along such axis; the steering radius is bounded. Also, a minimum distance between vehicles must be enforced along trajectories.

The task of each vehicle is to reach a given goal configuration from a given start configuration. Optimal solutions in the sense of minimizing total time will be considered.

The literature on optimal path planning for vehicles of this type is rather rich. The seminal work of Dubins [4] and the extension to vehicles that can back–up due to Reeds and Shepp [6], solved the single vehicle case by exploiting rather specialized tools. Later on, Sussmann and Tang [7], and Boissonnat et al. [2], reinterpreted these results as an application of Pontryagin's minimum principle [5]. Using these tools, Bui et al. [3] performed a complete optimal path synthesis for Dubins robots. The minimum principle framework is also fundamental in the developments presented here.

The paper is organized as follows. In section 2 we describe the problem and introduce some notation. In section 2.1 a formulation of the problem in a form amenable to application of optimal control theory is presented. Section 3 is devoted to the study of necessary conditions for extremal arcs. Finally, section 4 describes a numeric algorithm to find solutions, which applies under some restrictions.

2 Problem Statement

Consider N vehicles in the plane, whose individual configuration is described by $q_i = (x_i, y_i, \theta_i) \in \mathbf{R} \times \mathbf{R} \times S^1$, with (x_i, y_i) coordinates in a fixed reference frame (o, x, y) in the plane and θ_i the heading angle of the vehicle with respect to the x axis. Each vehicle is assigned a task, in order to compute its task a vehicle starts in a configuration $q_{i,s}$ and move in a final configuration $q_{i,g}$, we call this two particular configurations way–points. The initial way–point time is assigned and denoted by T_i^s. Assume vehicles are ordered such that $T_1^s \leq T_2^s \leq \cdots \leq T_N^s$. We denote by T_i^g the time at which the i–th vehicle reaches its goal, and let $T_i \overset{def}{=} T_i^g - T_i^s$. Motions of the i–th vehicle before T_i^s and after T_i^g are not of interest.

The i–th vehicle motion is subject to the constraint that its transverse velocity is zero, $\dot{x}_i \sin\theta_i - \dot{y}_i \cos\theta_i = 0$, $i = 1, \ldots, N$. Equivalently, this motion is described by the control system $\dot{q}_i = f_i(q_i, u_i, \omega_i)$, explicitly

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{pmatrix} = \begin{pmatrix} u_i \cos\theta_i \\ u_i \sin\theta_i \\ \omega_i \end{pmatrix}, \qquad (1)$$

where u_i and ω_i are the linear and angular velocity of the i–th vehicle, respectively. All vehicles are also supposed to be subject to the following additional constraints:

i) the linear velocity is constant: $u_i = \bar{u}_i$;

ii) the path curvature is bounded: $|\omega_i| \leq \Omega_i$, where $\Omega_i = \frac{\bar{u}_i}{R_i}$ and $R_i > 0$ denotes the minimum turning radius of the i–th vehicle;

iii) the distance between two vehicles must remain larger than, or equal to, a given separation limit: $D_{ij}(t) = (x_j(t) - x_i(t))^2 + (y_j(t) - y_i(t))^2 - d_{ij}^2 \geq 0$, at all times t ($d_{ii} = 0, i = 1, \ldots, N$).

We will consider problems in which the goal is to minimize the total execution time:

$$\left\{ \begin{array}{l} \min \sum_{i=1}^{N} T_i \\ \dot{q}_i = f_i(q_i, \bar{u}_i, \omega_i) \quad i = 1, \ldots, N \\ |\omega_i| \leq \frac{\bar{u}_i}{R_i} \quad i = 1, \ldots, N \\ D_{ij}(t) \geq 0, \quad \forall t, \ i, j = 1, \ldots, N \\ q_i(T_i^s) = q_{i,s}, \quad q_i(T_i^g) = q_{i,g}. \end{array} \right. \tag{2}$$

If separation constraints are disregarded, the minimum total time problem is clearly equivalent to N independent minimum length problems under the above constraints, i.e. to N classical Dubins' problems, for which solutions are well known in the literature ([4, 7, 2]). It should be noted that computation of the Dubins solution for any two given configurations is computationally very efficient.

2.1 Formulation as an Optimal Control Problem

Notice that the cost for the total time problem, $J = \sum_{i=1}^{N} T_i = \sum_{i=1}^{N} \int_{T_i^s}^{T_i^g} dt$, is not in the standard Bolza form. In order to use powerful results from optimal control theory, we rewrite the problem as follows. Let $h(t)$ denote the Heavyside function, i.e.

$$h(t) = \left\{ \begin{array}{ll} 0 & t < 0 \\ 1 & t \geq 0 \end{array} \right. ,$$

and define the window function $w_i(t) = h(t - T_i^s) - h(t - T_i^g)$. Then the minimum total time cost is written as

$$J = \int_0^{\infty} \sum_{i=1}^{N} w_i(t) dt \tag{3}$$

Using the notation $\mathrm{col}_{i=1}^N (v_i) = \left[v_1^T, \ldots, v_N^T\right]^T$, define the aggregated state $q = \mathrm{col}_{i=1}^N (q_i)$, controls $u = \mathrm{col}_{i=1}^N (\bar{u}_i)$ and $\omega = \mathrm{col}_{i=1}^N (\omega_i)$, and define the admissible control sets Ω accordingly. Also define the separation vector $D = [D_{12}, \cdots, D_{1N}, D_{23}, \cdots, D_{N-1,N}]$, and define the vector field $f(q, u, \omega) = \mathrm{col}_{i=1}^N (f_i w_i)$. Finally introduce matrices $\Gamma_i = \mathrm{col}_{j=1}^N (\sigma_{ij} [1 \ 1 \ 1]^T)$, with $\sigma_{ij} = 1$ if $i = j$, else $\sigma_{ij} = 0$, and functions $\gamma_i(q(t), \bar{q}) = \Gamma_i (q(t) - \bar{q})$. Our optimal control problem is then formulated as

Problem 1. *Minimize J subject to $\dot{q} = f(q, u, \omega)$, $\omega \in \Omega$, $D \geq 0$, and to the two sets of N interior–point constraints*

$$\begin{array}{ll} \gamma_i(q(t), q_i^s) = 0, & t = T_i^s \\ \gamma_i(q(t), q_i^g) = 0, & t = T_i^g (\text{unspecified}) \end{array}$$

3 Necessary conditions

Necessary conditions for problem 1 can be studied by adjoining the cost function with the constraints multiplied by unspecified Lagrange covectors. Omitting to write explicitly the extents of iterative operations when extending from 1 to N, let

$$\begin{array}{rl} \hat{J} = & \sum_i \pi_i^s \gamma_i(q(T_i^s) - q_i^s) \\ & + \sum_i \pi_i^g \gamma_i(q(T_i^g) - q_i^g) \\ & + \int_0^{\infty} \sum_i w_i + \lambda^T (\dot{q} - f) + \nu^T D \, dt, \end{array} \tag{4}$$

with λ and ν costates of suitable dimension, and with $\nu_i = 0$ if $D_i > 0$, $\nu_i \geq 0$ if $D_i = 0$. Let the Hamiltonian be defined as

$$H = \sum_i w_i + \lambda^T f + \nu^T D \tag{5}$$

Substituting 5 in 4, integrating by parts, and computing the variation of the cost, one gets:

$$\begin{array}{rl} \delta \hat{J} = & \sum_i \left[\lambda^T (T_i^{s-}) - \lambda^T (T_i^{s+}) + \pi_i^s \frac{\partial \gamma_i}{\partial q(T_i^s)}\right] d\, q(T_i^s) \\ & + \sum_i \left[\lambda^T (T_i^{g-}) - \lambda^T (T_i^{g+}) + \pi_i^g \frac{\partial \gamma_i}{\partial q(T_i^g)}\right] d\, q(T_i^g) \\ & + \sum_i \left[H(T_i^{g-}) - H(T_i^{g+}) + \pi_i^g \frac{\partial \gamma_i}{\partial T_i^g}\right] dT_i^g \\ & + \int_0^{\infty} \left[\left(\dot{\lambda}^T + \frac{\partial H}{\partial q}\right) \delta q + \frac{\partial H}{\partial \omega} \delta \omega\right] dt \end{array} \tag{6}$$

(recall that $dT_i^s \equiv 0$). Therefore, we have the following necessary conditions for an extremal solution:

$$\lambda_i(T_i^{s-}) = \lambda_i(T_i^{s+}) + \Gamma_i^T \pi_i^s \tag{7}$$

$$\lambda_i(T_i^{g-}) = \lambda_i(T_i^{g+}) + \Gamma_i^T \pi_i^g \tag{8}$$

$$H(T_i^{g-}) = H(T_i^{g+}) \quad (9)$$

$$\dot{\lambda}^T = -\frac{\partial H}{\partial q} \quad (10)$$

$$\frac{\partial H}{\partial \omega}\delta\omega = 0 \;\; \forall \delta\omega \text{admiss.} \quad (11)$$

Extremal trajectories for the i–th vehicle will be comprised in general of unconstrained arcs (with $D_{ij} > 0$, $\forall j \neq i$) and of constrained arcs, where the constraint is marginally satisfied ($\exists j : D_{ij} = 0$). We will proceed the discussion of necessary conditions by distinguishing constrained and unconstrained arcs.

3.1 Unconstrained arcs

Suppose that, for the i–th vehicle, the separation constraints are not active in the interior of an interval $[t_i^a, t_i^b]$, $T_i^s \leq t_i^a < t_i^b \leq T_i^g$, i.e. $D_{ij}(t) > 0$, $j = 1, \ldots, N$, $t \in (t_i^a, t_i^b)$. Expanding 10, one gets

$$\left[\dot{\lambda}_{i1}, \dot{\lambda}_{i2}, \dot{\lambda}_{i3}\right] = [0, 0, \lambda_{i,1}\bar{u}_i \sin\theta_i - \lambda_{i2}\bar{u}_i \cos\theta_i]\,. \quad (12)$$

The characterization of optimal solutions in the unconstrained case proceeds along the lines of the classical Dubins solution (see [4, 7, 2]). We report some results here for reader's convenience. By integrating 12 one gets $\lambda_{i1}(t_i^a < t < t_i^b) = \bar{\lambda}_{i1}$, $\lambda_{i2}(t_i^a < t < t_i^b) = \bar{\lambda}_{i2}$, and $\lambda_{i3}(t_i^a < t < t_i^b) = \bar{\lambda}_{i1} y_i(t) - \bar{\lambda}_{i2} x_i(t) + \bar{\lambda}_{i3}$, with constant $\bar{\lambda}_{i,j}$, $j = 1, 2, 3$. In light of these relationships, conditions 7 and 8 state that the costate components λ_{i1} and λ_{i2} are piecewise constant, with jumps possibly at the start and arrival time of the i–th vehicle. The addend in the Hamiltonian relative to the i–th vehicle can be written as $H_i = 1 + \bar{u}_i \rho_i \cos(\theta_i - \psi_i) + \lambda_{i3}\bar{u}_i\omega_i$, where $\rho_i = \sqrt{\bar{\lambda}_{i1}^2 + \bar{\lambda}_{i2}^2}$ and $\psi_i = \text{atan2}\,(\bar{\lambda}_{i2}, \bar{\lambda}_{i1})$. From Pontryagin's Minimum Principle (PMP), we know that $H_i(t) = \mathit{const.} \leq 0$ along extremal unconstrained arcs and, being by assumption the way-points configurations unconstrained, it follows from 9 that $H_i(t)$ is also continuous at $t = T_i^g$.

Extremals of H_i within the open segment $\{|\omega_i| < \bar{u}_i/R\}$ can only obtain if

$$\frac{\partial H_i}{\partial \omega_i} = \lambda_{i3} = \bar{\lambda}_{i1} y_i(t) - \bar{\lambda}_{i2} x_i(t) + \bar{\lambda}_{i3} = 0. \quad (13)$$

If the condition holds on a time interval of nonzero measure, then $\dot{\lambda}_{i,3} = 0$ on the interval: this implies $\rho_i \bar{u}_i \sin(\theta_i - \psi_i) = 0$, hence $\theta_i = \psi_i \bmod \pi$ and $\omega_i = 0$. In such an interval, the vehicle moves on the straight route (*the supporting line*) in the horizontal x, y plane described in 13. Other extremals of H_i occur at $\omega = \pm\bar{u}_i/R$. The sign of the minimizing yaw rate ω_i is opposite to that of λ_{i3}; in other words, the supporting line also represent the switching locus for the yaw rate input. Trajectories corresponding to $\omega_i = \pm\bar{u}_i/R$ correspond to circles of minimum radius R followed counterclockwise or clockwise, respectively. It is important for our further developments to notice that, along extremal arcs, also the costates are completely determined by boundary configurations up to a multiplicative constant $\rho \neq 0$, which remains undetermined.

For each vehicle, extremal unconstrained arcs are concatenations of only two types of elementary arcs: line segments of the supporting line (denoted as "S"), and circular arcs of minimum radius (denoted by "C"). The latter type can be further distinguished between "R" clockwise arcs ($\omega_i = \bar{u}_i/R$), and "L" counterclockwise arcs ($\omega_i = -\bar{u}_i/R$). According to the widespread usage, subscripts will be used to denote the length of rectilinear segments, and the angular span of circular arcs.

Switchings of ω_i among 0, $\bar{u}_i/R$, and $-\bar{u}_i/R$ can only occur when the vehicle center is on the supporting line. As a consequence, all extremal unconstrained paths of each vehicle are written as $C_{u_1} S_{d_1} C_{u_2} S_{d_2} \cdots S_{d_n} C_{u_n}$, with $u_i = 2k\pi$, k integer, $i = 2, \ldots, n-1$.

In the case of a single vehicle, the discussion of optimal unconstrained arcs can be further refined by several geometric arguments, for which the reader is referred directly to the literature [4, 7, 2]. Optimal paths necessarily belong to either of two path types in the Dubins' sufficient family:

$$\{C_a C_b C_e\ ,\ C_u S_d C_v\} \quad (14)$$

with the restriction that

$$b \in (\pi R, 2\pi R);\;\; a, e \in [0, b],\;\; u, v \in [0, 2\pi R),\;\; d \geq 0 \quad (15)$$

A complete synthesis of optimal paths for a single Dubins vehicle is reported in [3]. The length of Dubins paths between two configurations, denoted by $L_D(\xi_i^s, \xi_i^g)$, is then unique and defines a metric on $\mathbb{R}^2 \times S^1$. One simply has $L_D(\cdot,\cdot) = R(|a| + |b| + |c|)$ for a $C_a C_b C_e$ path, and $L_D(\cdot,\cdot) = R(|u| + |v|) + d$ for a $C_u S_d C_v$ path.

In our multivehicle problem, however, other extremal paths may turn out to be optimal, and therefore have to be considered. This may happen for instance for a path of type $C_a S_b C_{2k\pi} S_e C_f$ if (and only if) the corresponding Dubins' path $C_a S_{b+e} C_f$, which is shorter, is not collision free. Arcs of type $C_{2k\pi}$ can be interpreted as waiting–in–circles maneuver for another vehicle to pass by and avoid collision (compare e.g. with current practice in conflict resolution for air traffic control). Notice explicitly that the length of two subpaths of type $\cdots C_{u_i} S_\alpha C_{2k\pi} S_\beta C_{u_{i+1}} \cdots$ and $\cdots C_{u_i} S_\gamma C_{2k\pi} S_\delta C_{u_{i+1}} \cdots$ are equivalent as far as $\alpha + \beta = \gamma + \delta$.

By "extremal trajectory" (Dubins' trajectory, respectively) we indicate henceforth a map $\mathbb{R}^+ \mapsto \mathbb{R}^2$ defined by $\left(x_i^D(t), y_i^D(t)\right)$, denoting the position of the i–th vehicle at time t along an extremal (Dubins') path connecting q_i^s to q_i^g.

Remark 1. If a set of non–colliding Dubins' trajectories exists, then this is obviously a solution of the minimum total time problem. More interestingly, if with all combinations of possible independent Dubins trajectories a collision results, then the optimal solution will contain at least a constrained arc or at least one wait circle.

3.2 Constrained arcs

Some further manipulation of the cost function is instrumental to deal with constrained arcs, i.e. arcs in which at least two vehicles are exactly at the critical separation ($D_{ij} = 0$, $i \neq j$). To fix some ideas, let us consider a constrained arc involving only vehicles 1 and 2. Along a constrained arc, the derivatives of the constraint must vanish:

$$N = \begin{bmatrix} D_{12} \\ \dot{D}_{12} \end{bmatrix} = \begin{bmatrix} (x_2 - x_1)^2 + (y_2 - y_1)^2 - d^2 \\ 2(x_2 - x_1)(\dot{x}_2 - \dot{x}_1) + 2(y_2 - y_1)(\dot{y}_2 - \dot{y}_1) \end{bmatrix} = 0 \tag{16}$$

with $d = d_{12}$ Let ϕ be the direction of the segment joining the two vehicles, so that

$$\begin{aligned} x_2 - x_1 &= d\cos\phi, \\ y_2 - y_1 &= d\sin\phi, \end{aligned} \tag{17}$$

From the second equation in 16, one gets

$$(x_2 - x_1)(\bar{u}_2\cos\theta_2 - \bar{u}_1\cos\theta_1) + (y_2 - y_1)(\bar{u}_2\sin\theta_2 - \bar{u}_1\sin\theta_1) = 0 \tag{18}$$

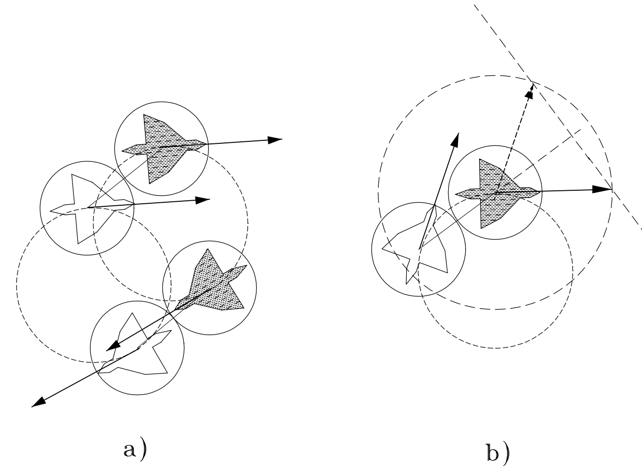

Figure 1: *Possible constrained arcs for two vehicles with the same velocity*

and, using 17,

$$\bar{u}_1\cos(\phi - \theta_1) - \bar{u}_2\cos(\phi - \theta_2) = 0. \tag{19}$$

When the constraint is active, the two vehicle envelopes are in contact, and the relative orientation of the two vehicles must satisfy 19, which defines (for given $\bar{u}_1, \bar{u}_2$) two manifolds of solutions in the space $\{(\theta_1, \theta_2, \phi) \in S^1 \times S^1 \times S^1\}$ described as

$$\textbf{a)} \quad \theta_2^a = \phi + \arccos\left(\frac{\bar{u}_1}{\bar{u}_2}\cos(\phi - \theta_1)\right); \tag{20}$$

$$\textbf{b)} \quad \theta_2^b = \phi - \arccos\left(\frac{\bar{u}_1}{\bar{u}_2}\cos(\phi - \theta_1)\right). \tag{21}$$

The two solutions correspond to two different types ("a" and "b") of relative configurations in contact. For instance, for $\bar{u}_1 = \bar{u}_2$, one has:

$$\textbf{a)} \quad \theta_2^a = \theta_1; \tag{22}$$

$$\textbf{b)} \quad \theta_2^b = 2\phi - \theta_1. \tag{23}$$

In case a) the two vehicles have the same direction, while in case b) directions are symmetric with respect to the segment joining the vehicles (see 1).

The two solutions 20, 21 coincide for

$$\phi = \theta_1 \pm \arccos\left(\frac{\bar{u}_2}{\bar{u}_1}\right), \tag{24}$$

such a ϕ exists only if $\frac{\bar{u}_2}{\bar{u}_1} \leq 1$. If we find the solution of 19 in θ_1^a and θ_1^b, the solutions coincide for

$$\phi = \theta_2 \pm \arccos\left(\frac{\bar{u}_1}{\bar{u}_2}\right), \tag{25}$$

ϕ exists if $\frac{\bar{u}_1}{\bar{u}_2} \leq 1$. Hence, from 24 and 25, ϕ exists if $\bar{u}_1 = \bar{u}_2$ and in this case the solutions of 19 is

$$\phi = \theta_1 = \theta_2.$$

In order to study constrained arcs of extremal solutions, it is useful to rewrite the cost function 4 as

$$\begin{aligned}\bar{J} = \;& \beta^T N \\ & + \textstyle\sum_i \pi_i^s \gamma_i(q(T_i^s) - q_i^s) \\ & + \textstyle\sum_i \pi_i^g \gamma_i(q(T_i^g) - q_i^g) \\ & + \textstyle\int_0^\infty \sum_i w_i + \lambda^T(\dot{q} - f) + \mu \ddot{D}_{12} dt,\end{aligned} \tag{26}$$

with $\mu \geq 0$ along a constrained arc. The jump conditions at the entry point of a constrained arc, occurring at time τ, are now

$$\lambda_i(\tau^-) = \lambda_i(\tau^+) + \beta \left.\frac{\partial N}{\partial q}\right|_\tau \tag{27}$$

$$H(\tau^-) = H(\tau^+) \tag{28}$$

$$\tag{29}$$

where $H = \sum_i w_i + \lambda^T f + \nu^T \ddot{D}_{12}$, and

$$\left(\frac{\partial N}{\partial q}\right)^T = 2\begin{bmatrix} (x_1 - x_2) & \bar{u}_1 \cos\theta_1 - \bar{u}_2 \cos\theta_2 \\ (y_1 - y_2) & \bar{u}_1 \sin\theta_1 - \bar{u}_2 \sin\theta_2 \\ 0 & d\bar{u}_1 \sin(\phi - \theta_1) \\ (x_2 - x_1) & \bar{u}_2 \cos\theta_2 - \bar{u}_1 \cos\theta_1 \\ (y_2 - y_1) & \bar{u}_2 \sin\theta_2 - \bar{u}_1 \sin\theta_1 \\ 0 & -d\bar{u}_2 \sin(\phi - \theta_2) \end{bmatrix}.$$

A further distinction among constrained arcs of zero and nonzero length should be done at this point.

3.2.1 Constrained arcs of zero length

Consider first a constrained arc of zero length occurring at a generic contact configuration, which is completely described by the configuration of one vehicle (e.g., $q_c = q_1$), by the angle $\phi_c = \phi$, and by the contact type. Assume for the moment that there is only one constrained arc of zero length in the optimal path between way-points of the two vehicles. Equation 27, taking into account that costates of each vehicle are determined (once the way-points and contact configurations are fixed) up to constants $\rho_i(\tau^-)$, $\rho_i(\tau^+)$, provides a system of 6 equations in 6 unknowns of the form

$$A(q_c, \phi_c)\begin{bmatrix} \rho_1(\tau^-) \\ \rho_1(\tau^+) \\ \rho_2(\tau^-) \\ \rho_2(\tau^+) \\ \beta_1 \\ \beta_2 \end{bmatrix} = 0,$$

where the explicit expression of matrix $A(q_c, \phi_c)$, for each contact type, can be easily evaluated in terms of q_1^s, q_1^g, q_2^s, q_2^g, and is omitted here for space limitations. Non–triviality of costates implies that (q_c, ϕ_c) must satisfy $\det(\mathbf{A}) = 0$. A further constraint on contact configurations is implied by the equality of deplacement times from start to contact for the vehicles, which is expressed in terms of Dubins distances as

$$L_D(\xi_1^s, \xi_c)/\bar{u}_1 = L_D(\xi_2^s, \xi_c')/u_2,$$

where ξ_c' denotes the configuration of vehicle 2 at contact, which is uniquely determined for each contact type. If m constrained arcs of zero length are present

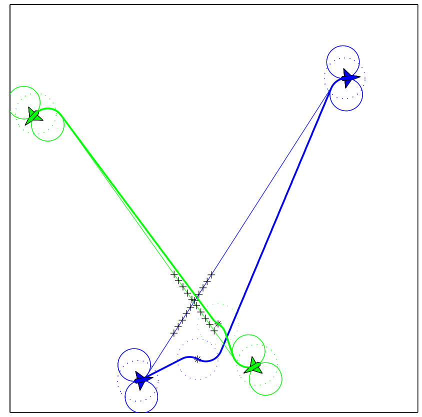

Figure 2: *A numerically computed solution to a two–vehicles minimum total time problem. Vehicles are represented as aircraft. Minimum curvature circles are reported at the start and goal configurations, along with safety discs of radius $d/2$ (dashed). The unconstrained Dubins' paths (thin lines) would achieve a cost of 88.75 units, but collide in this case (collisions are marked by "+" signs). The optimal solution consists of two unconstrained arcs for each vehicle, pieced together with a zero–length constrained arc of type b. Total cost is 92.25 units.*

in an optimal solution, similar conditions apply (with way-points configurations suitably replaced by previous or successive contact configurations), yielding $2m$ equations in $4m$ unknowns.

3.2.2 Constrained arcs of nonzero length

From this point on, we will make the assumption that forward velocity of all vehicles are equal to 1 ($\bar{u}_i = 1$). Consider an interval $[T_1, T_2]$ during which the constraint $D_{12} \equiv 0$. [1] A configuration of the two vehicles along such constrained arcs can be completely described by using only four parameters, for instance the configuration (x_1, y_1, θ_1) of one vehicle and the value of ϕ. In fact, due to the tangency conditions on the constraint, one has 17 and either 22 or 23. Moreover, differentiating 17, one finds

$$\begin{aligned} \dot{x}_2 &= \dot{x}_1 - d\dot{\phi}\sin\phi, \\ \dot{y}_2 &= \dot{y}_1 + d\dot{\phi}\cos\phi, \end{aligned} \tag{30}$$

and

$$\dot{\phi} = \frac{1}{d}[\sin(\theta_2 - \phi) - \sin(\theta_1 - \phi)]. \tag{31}$$

Differentiating twice $D_{1,2}$ we obtain:

$$\begin{aligned} &\ddot{D}_{12}(q, \omega, t) = 0 = \\ &4 - 4\cos(\theta_1 - \theta_2) + 2\omega_1 d\sin(\theta_1 - \phi) - 2\omega_2 d\sin(\theta_2 - \phi). \end{aligned} \tag{32}$$

Constrained arcs of nonzero length that are part of an optimal solution must themselves satisfy necessary conditions, which can be deduced by rewriting the problem in terms of the reduced set of variables.

$$\begin{cases} \min 2(T_2 - T_1) \\ \dot{x}_i = \cos\theta_i \\ \dot{y}_i = \sin\theta_i \\ \dot{\theta}_i = \omega_i \\ \dot{\phi} = \frac{1}{d}[\sin(\theta_2 - \phi) - \sin(\theta_1 - \phi)] \\ \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \end{cases} \tag{33}$$

for $i = 1$ or $i = 2$, and for some initial and final specification of the variables $(x_i, y_i, \theta_i, \phi)$ and of the constrained arc type (a or b). Recall that $\theta_2 = \theta_1$ (arcs of type a), or $\theta_2 = 2\phi - \theta_1$ (type b).

[1] It should be pointed out that the study of constrained arcs of nonzero length is also useful to model cooperative manipulation of object by multiple vehicles, assuming that each vehicle supports the common load through a hinge joint.

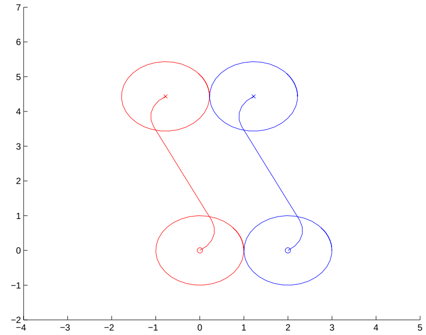

Figure 3: *Extremal constrained arcs of type a consist of two copies of a Dubins' path*

Let us consider the two types of constrained arcs separately. Notice that two extremal constrained arcs of different type may be pieced together through a configuration with $\theta_1 = \theta_2 = \phi$, which is both of type a and b. **Type a).** From 31, $\phi(t) \equiv \phi_0 = \arctan\frac{y_2(0) - y_1(0)}{x_2(0) - x_1(0)}$, hence

$$\begin{aligned} \dot{x}_1 &= \dot{x}_2 \\ \dot{y}_1 &= \dot{y}_2 \\ \omega_1 &= \omega_2 \end{aligned} \tag{34}$$

Extremal constrained arcs of type a consist of a Dubins path for vehicle 1, and of a copy of the same path translated in the plane by $[d\cos\phi_0, d\sin\phi_0]^T$ for the other vehicle (see 3).

Type b). In this case, using 32, one obtains $\dot{\phi} = \frac{1}{2}(\omega_1 + \omega_2)$ in 33. Introduce $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, and $H = 2 + \lambda_1\cos\theta_1 + \lambda_2\sin\theta_1 + \lambda_3\omega_1 + \lambda_4(\omega_1 + \omega_2)/2$. Necessary conditions for optimality of solutions of 33 are

$$\dot{\Lambda} = -\Lambda\begin{pmatrix} 0 & 0 & -\sin\theta_1 & 0 \\ 0 & 0 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{35}$$

Hence, λ_1, λ_2 and λ_4 are constant. Letting $\lambda_1 = \rho\cos\psi$ and $\lambda_2 = \rho\sin\psi$, from 35 one gets

$$\dot{\lambda}_3 = \rho\sin(\theta_1 - \psi). \tag{36}$$

From P.M.P one also gets that, when $|\omega_1| < \Omega_1$ and $|\omega_2| < \Omega_2$, it is necessary for an optimal arc that $\frac{\partial H}{\partial\omega_1} = \frac{\partial H}{\partial\omega_2} = 0$, which implies $\lambda_3 = \lambda_4 = 0$. In this case, from $\dot{\lambda}_3 = 0$ one easily gets $\theta_1 = \psi \pm \pi$, $\omega_1 = 0$. The direction

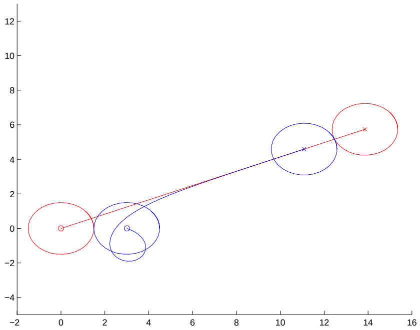

Figure 4: *Singular extremals in a constrained arc of type b.*

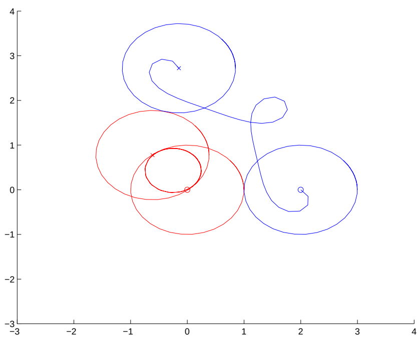

Figure 5: *An extremal constrained arc of type b.*

of the segment joining the two vehicles varies as

$$\begin{array}{l} \dot{\phi} = \frac{2}{d}\sin(\phi - \theta_1) \\ \phi(0) = \phi_0 \end{array} \quad (37)$$

Equilibria of 37 at $\phi = \theta_1$ and $\phi = \theta_1 - \pi$ are respectively unstable and asymptotically stable. Hence, along a singular constrained arc of type b, one vehicle will be moving on a straight line, while the other will be trailing behind (see 4).

Extremal constrained arcs may also obtain when a control variable is on the border of its domain, e.g. $\omega_1 = \pm\Omega_1$. In this case the motion of the two vehicles result in arcs such as those represented in figure 5.

Notice that along such an arc, the steering velocity of vehicle 2 is uniquely determined by 32. Hence, being $|\omega_2|$ bounded by Ω_2, an arc of maximum curvature for vehicle one will be possible only until the limit curvature for vehicle two is reached. In other words, letting

$$m_1(q) = \max\{-\Omega_2 - \frac{4}{d}\sin(\theta_1 - \phi), -\Omega_1\},$$

and

$$M_1(q) = \min\{\Omega_2 - \frac{4}{d}\sin(\theta_1 - \phi), \Omega_1\},$$

we have that along constrained arcs of type b we must have $m_1 \leq \omega_1 \leq M_1$. Control ω_1 may equal M_1 if $\omega_1 = \Omega_1$ or if $\omega_1 = \Omega_2 - \frac{4}{d}\sin(\theta_1 - \phi)$. In the latter case, $\omega_2 = -\Omega_2$. A similar reasoning applies to vehicle two, for which we get $m_2 \leq \omega_2 \leq M_2$. In conclusion, along a nonsingular extremal constrained arc of type b, one of the vehicles moves along a circle of minimum radius, while the other follows a curve such as that described in figure 5.

4 Numerical computation of solutions

The necessary conditions studied in this paper provide useful hints in the search for an optimal solution to the problem of planning trajectories of N vehicles in a common workspace. Although a complete synthesis has not been obtained so far, we will describe in this section an algorithm that finds efficient solutions to the optimal planning problem in a reasonably short time.

Based on the discussion above, the optimal conflict resolution paths for multiple vehicles may include multiple waiting circles and constrained arcs of both zero and nonzero length. The algorithm to be described shortly was developed to solve air traffic control problems [1], and is based on a few simplifying assumptions motivated by the particular application. Namely, we asssume henceforth that

h1 all vehicles have equal geometric characteristics and equal (constant) speed;

h2 constrained arcs of nonzero length are not considered;

h3 multiple zero–length constrained arcs among the same vehicles are ruled out;

h4 the initial configurations of the vehicles are sufficiently separated.

With assumption **h4** we mean that for each vehicle, the initial configuration are collision free and guarantee that wait circles at the initial configuration are collision free (this holds for instance if the distance between the initial position of vehicles i and j is larger than $2\pi R\frac{\bar{u}_j}{\bar{u}_i} + 2R + \frac{d_{ij}}{2}$).

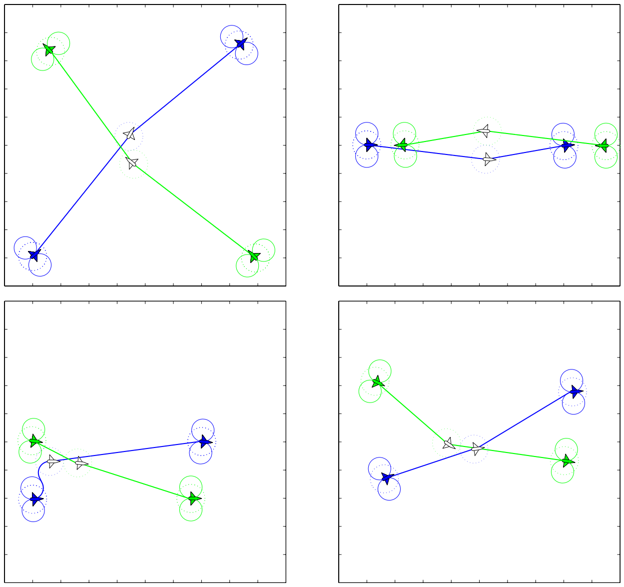

Figure 6: *Numerically computed solutions to optimal cooperative conflict resolution for two vehicles. Minimum curvature circles are reported at the way-point configurations, along with safety discs of radius $d/2$ (dashed). Optimal solutions consist of two unconstrained Dubins' trajectories for each vehicle, pieced together with a zero–length constrained arc.*

Consider first the case of two vehicles. If the Dubins' trajectories joining the way-points configurations do not collide (i.e., $D(t) \geq 0, \forall t$), this is the optimal solution. Otherwise we compute the shortest contact–free solution with wait circles at the initial configurations, and let its length be L_f.

Hence we look for a solution with a concatenation of two Dubins' paths and a single constrained zero–length arc of either type a) or b) for both vehicles. Such solution can be searched over a 2–dimensional submanifold of the contact configuration space ($\mathbb{R}^2 \times S^1 \times S^1$). The optimal solution can be obtained by using any of several available numerical constrained optimization routines: computation is sped up considerably by using very efficient algorithms made available for evaluating Dubins' paths ([3]). The lenght L_c of such solution is compared with L_f, and the shorter solution is retained as the two–vehicle optimal conflict management path with at most a single constrained zero-length arc (OCMP21, for short). Some examples of OCMP21 solutions are reported in Figure 6.

If N vehicles move in a shared workspace, their possible conflicts can be managed with the following multilevel policy:

Level 0 Consider the unconstrained Dubins paths of all vehicles, which may be regarded as N single–vehicle, optimal conflict management paths, or OCMP10. If no collision occurs, the global optimum is achieved, and the algorithm stops. Otherwise compute the shortest contact–free paths (with wait circles) and go to next level;

Level 1 Consider the $M = 2 \binom{N}{2}$ possible solutions with a single contact (of either type a or b), between two vehicles, and possibly wait circles for other vehicles, and compute the shortest path in this class. If this is longer than the shortest path obtained at level 0, exit. Otherwise, continue;

Level $m \geq 2$. Consider the $M \prod_{\ell=1}^{m-1} (M - 2^\ell)$ possible solutions involving m zero–length constrained arcs between different pairs of vehicles and (possibly) wait circles for other vehicles, and compute the shortest path in this class. If this is longer than the shortest path obtained at level $m - 1$, exit. Otherwise, continue;

A few three–vehicle conflict resolution trajectories at different levels are reported in Figure 7. When the number of vehicles increases, the number of optimization problems to be solved grows combinatorially. However, in practice, it is hardly to be expected that conflicts between more than a few vehicles at a time have to be managed. It was also observed in our simulations that, for vehicle parameters close to those of the kinematic model of commercial aircraft, solutions including wait circles are very rare.

5 Conclusion

In this paper, we have studied the problem of planning trajectories of multiple Dubins' vehicles in a plane. Necessary conditions have been derived for both free and constrained arcs. An algorithm for numerically finding solutions has been described.

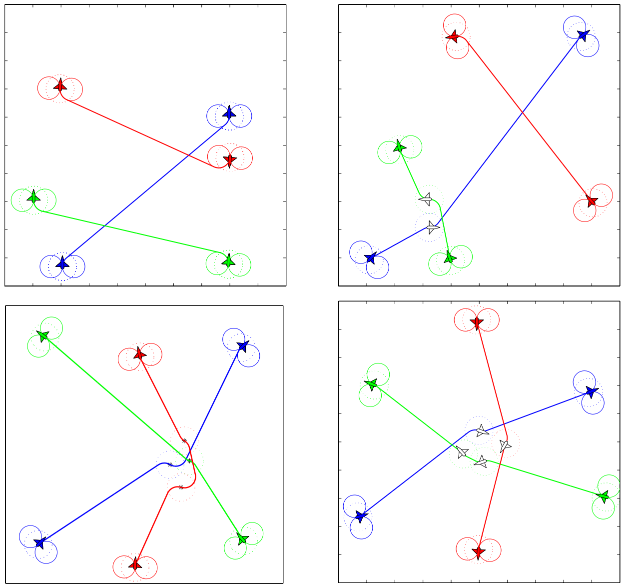

Figure 7: *Four cases of three–vehicles conflict resolution. Up left: the conflict is resolved at level 0. Up-right: a level 1 solution. Down left: a level 2 solution whereby the vehicle starting in the middle contacts first the one arriving on its right, and after the one arriving from left. Down right: a level 2 resolution that generates a roundabout–like maneuver.*

Future work on this topic will address the problem of finding a complete optimal synthesis at least for the simplest cases ($N = 2$). Further refinement of the algorithm will be sought, that could exploit more of the rich structure optimal solutions must satisfy. Finally, optimal paths of multiple agents at fixed distance will be studied in more detail, to address such problems as cooperative manipulation of objects by robotic vehicles and formation flight planning.

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