Controllability Properties for Aircraft Formations

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***Abstract*— This paper studies the controllability of formations of n identical aircraft maintaining constant distances. Aircraft are modeled as a planar kinematic system with constant velocity and curvature bounds. The challenges of achieving controllability of such system are that it is an affine system with drift and its admissible controls are determined by its configuration variables. We begin with the study of a pair of aircraft maintaining a constant distance. As a result, we show that if the specified distance is sufficiently large, a pair of aircraft is completely controllable, i.e. can be steered between any two arbitrary configurations. In case of small distances, a description of the reachable sets is provided. Finally, we provide the controllability results for three basic formations of n aircraft.**

I. Introduction

This paper provides the results of controllability for systems of identical aircraft maintaining a constant distance, i.e., a formation of aircraft. Controllability results are fundamental for the design of the motion planning of formation of aircraft and the existence and study of optimal formation trajectories.

The formation problem for multiple robots has been extensively studied in the past. Different challenges have been exploited and different approaches have been proposed, see e.g. [1] and references therein. The importance of formation flight is based on the advantages that the formation provides such as the reduction of the fuel consumption by decreasing turbulence. Motivations and applications of the formation problem come also from search, rescue and security patrol. In the literature, several aspects of the formations have been taken into account such as the stability and the maintenance of the formation. However, to the authors' best knowledge, the controllability properties have not been determined.

A system is *controllable* if, for every pair of points p and q in the configuration space, there exists a control that steers the system from p to q (see [2],[3]). The controllability of a system answers the question about the existence of an admissible trajectory between given any two configurations, which is an important condition for a feasible design of motion planning ([4]) and for the existence of an optimal trajectory (see e.g. [2]). Moreover, the study of pairs of vehicles maintaining a constant distance helps the design of navigation strategies for a group of aircraft moving in formation (see e.g. [5], [6] and [7]).

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Aircraft that cruise within a given altitude layer can be modeled as a kinematic system with constant velocity and curvature bounds, i.e. as Dubins vehicles, [8]. In robotics there are five most common types of robot vehicles: Dubins [9], Reeds-Shepp (RS) [10], differential drive (DDV) [11], [12], car-like (Car) [13], [14] and convexified Reeds-Shepp (CRS) [2]. Compared with the other four types of vehicles, whose controllability properties were discussed in [15], the system for pairs of Dubins vehicles is not symmetric and it is an affine control system with drift. Moreover, the ranges of its admissible controls are determined by its configuration variables. Hence, the controllability for Dubins vehicles is the most complex one. We apply a geometric method to prove that it is a weakly reversible system, and then use the accessibility rank condition to prove its controllability.

Controllability results for pairs of Dubins vehicles maintaining costant distances are discussed in three cases in terms of relationships between the distance D to be maintained and the minimal turning radius $R_{\min}$: $D = 2R_{\min}$; $D < 2R_{\min}$ and $D > 2R_{\min}$. In each case, the configuration space and reachable set are provided. As a result, within the defined configuration space, the reachable sets consist of two independent subsets when $D \leq 2R_{\min}$. On the other hand, the system is controllable within the defined configuration space when $D > 2R_{\min}$.

Finally, the controllability results provide a direct extension for the controllability of formations of n identical aircraft for $D > 2R_{\min}$. In section VI, controllability properties for star, chained and ring formations are reported.

II. Kinematic model and subsystems

We first study the controllability properties for a pair of identical aircraft maintaining a constant distance D. In [8] it has been shown how aircraft, cruising at a constant altitude, can be modeled as a kinematic system with constant velocity and curvature bounds. Under those assumptions aircraft can be modeled as Dubins vehicles. Hence, let $(x_i, y_i, \theta_i) \in \mathbb{R}^2 \times \mathcal{S}^1$, $i = 1, 2$ denotes a configuration of Dubins aircraft i, where (x_i, y_i) is the position of aircraft i and θ_i denotes the forward direction angle of aircraft i with respect to the positive x-axis. Without loss of generality we can assume the minimum turning radius $R_{\min} = 1$, although in order to emphasize its influence R_{min} often remains.

Let $\bar{x} = (x_1, y_1, \theta_1, x_2, y_2, \theta_2) \in \mathbb{R}^2 \times \mathcal{S}^1 \times \mathbb{R}^2 \times \mathcal{S}^1$ be a configuration of a pair of aircrafts, the kinematic model of system is:

$$\dot{\bar{x}} = (\cos\theta_1,\, \sin\theta_1,\, v_1,\, \cos\theta_2,\, \sin\theta_2,\, v_2)^T \tag{1}$$

subject to the constant distance constraint

$$d(t) = (y_2(t) - y_1(t))^2 + (x_2(t) - x_1(t))^2 = D^2. \tag{2}$$

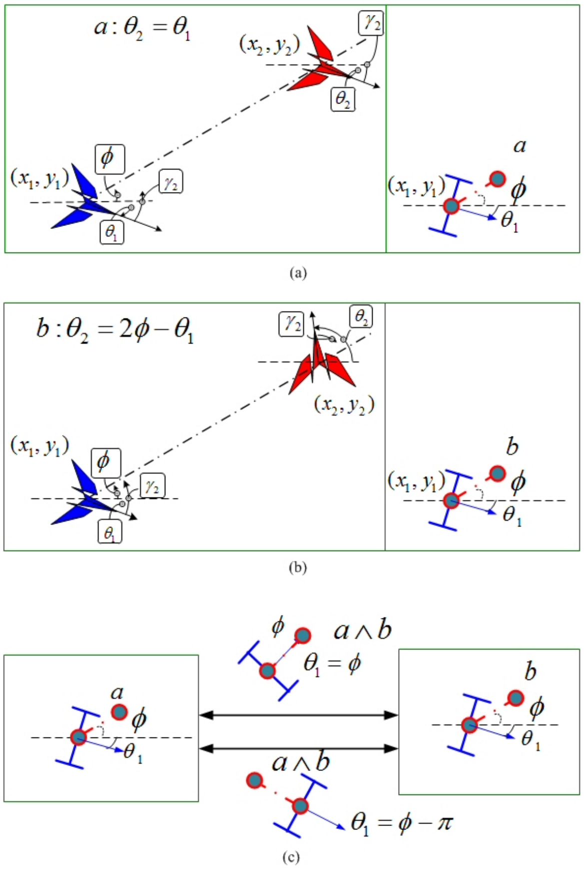

Fig. 1. For given $(x_1, y_1, \theta_1, \phi)$, the configuration of aircraft 2 is determined with two possible angles $a : \theta_2 = \theta_1$ in (a) and $b : \theta_2 = 2\phi - \theta_1$ in (b). In the right side of (a) and (b), configurations are represented by 4-dimensional parameters plus two possible angular relations a and b. Their combined cases are shown in (c).

Let Σ_D denote the system (1) with constraint (2). And for each aircraft, its admissible control (angular velocity) is $v_i \in U = [-1, 1]$.

Given the distance constraint (2), another possible configuration of Σ_D is thus $q = (x_1, y_1, \theta_1, \phi, \theta_2)$ where ϕ is the angle of vector $(x_2 - x_1,\ y_2 - y_1) = (D\cos\phi,\ D\sin\phi)$ with respect to the x-axis, illustrated in fig.1. As shown in [8], in order to maintain a constant distance ($\dot{d}(t) = 0$), either one of two angular relationships must hold:

$$a : \theta_1 = \theta_2;\ b : \theta_1 + \theta_2 = 2\phi. \tag{3}$$

Once $(x_1, y_1, \theta_1, \phi)$ is given, the configuration $(x_2,\ y_2,\ \theta_2)$ can be obtained as follows: $x_2 = D\cos\phi + x_1$, $y_2 = D\sin\phi + y_1$ and $\theta_2 = \theta_1$ or $\theta_2 = 2\phi - \theta_1$, shown in fig.1 (a) and (b). The combined cases for angular relation a and b is $a \wedge b : \theta_1 = \phi \vee \theta_1 = \phi - \pi$, shown in fig.1 (c). For simplicity and clarity of representation, the 4-dimensional parameters $(x_1, y_1, \theta_1, \phi)$ plus one of the two possible conditions, a or b, are introduced as shown in the right side of fig.1.

Let Σ_D^a denote the system Σ_D when relation a holds. Notice that $\theta_1 \equiv \theta_2$ implies $v_1 = v_2$. From the definition of ϕ, when a holds, we also have $\dot{\phi} = 0$.

Thus, the kinematic model of Σ_D^a can be written as:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\theta}_1 \\ \dot{\phi} \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} v_1 \tag{4}$$

with $v_1 \in [-1, 1]$.

Let Σ_D^b be the system Σ_D when relation b holds, in which $\theta_2 = 2\phi - \theta_1$ now implies $v_2 = 2\dot{\phi} - v_1$.

From the definition of ϕ and the vehicles kinematics we obtain

$$v_1 + v_2 = 2\dot{\phi} = \frac{4\sin(\phi - \theta_1)}{D} \tag{5}$$

Finally, accordingly to (5) and $-1 \le v_1, v_2 \le 1$, we can derive $v_1 \in U_1$ with:

$$U_1 = [\max\{-1, \frac{4\sin(\phi - \theta_1)}{D} - 1\}, \min\{1 + \frac{4\sin(\phi - \theta_1)}{D}, 1\}]. \tag{6}$$

Thus, the kinematic model of Σ_D^b can be written as:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\theta}_1 \\ \dot{\phi} \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \\ \frac{2\sin(\phi - \theta_1)}{D} \\ \frac{4\sin(\phi - \theta_1)}{D} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} v_1 \tag{7}$$

where $v_1 \in U_1$.

Notice that the distance parameter D influences the evolution of the system, according to the kinematic model of Σ_D^b (7), hence , it also influences the configuration space and the reachable set of system Σ_D. Therefore, we will analyze the controllability of the system in three cases in terms of the relationships between D and $R_{\min}$:

1) $D - 2R_{\min} = 0$: this case is denoted as Σ_{D^0};
2) $D - 2R_{\min} < 0$: this case is denoted as Σ_{D^-};
3) $D - 2R_{\min} > 0$: this case is denoted as Σ_{D^+}.

III. CONTROLLABILITY FOR Σ_{D^0}

We start focusing on the particular case $D = 2R_{\min}$.

Let $\mathcal{S}^a$ be a subset of $\mathcal{S}^1$ with angular relation constraint a holding and $\mathcal{S}^b$ be a subset of $\mathcal{S}^1$ with angular relation constraint b holding. Let $\mathcal{M}^a = \mathbb{R}^2 \times \mathcal{S}^1 \times \mathcal{S}^1 \times \mathcal{S}^a$ and $\mathcal{M}_{D^0}^b = \mathbb{R}^2 \times \mathcal{S}^1 \times \mathcal{S}^1 \times \mathcal{S}^b$.

Remark 1: (Configuration Space of System Σ_{D^0}) For $D = 2R_{\min}$, let $(x_1, y_1, \theta_1, \phi, \theta_2)$ be a configuration of Σ_{D^0}. The configuration space of Σ_{D^0} is $\mathcal{M}_{D^0} = \mathcal{M}^a \cup \mathcal{M}_{D^0}^b$.

For simplicity, in the rest of this section, we omit the subscript D^0.

Let $\mathcal{R}(q)$ denote the set of points reachable from q. For Σ from $q^s = (x_1^s, y_1^s, \theta_1^s, \phi^s, \theta_2^s)$, we will characterize $\mathcal{R}(q^s)$ by investigating evolutions of the subsystems Σ^a and Σ^b.

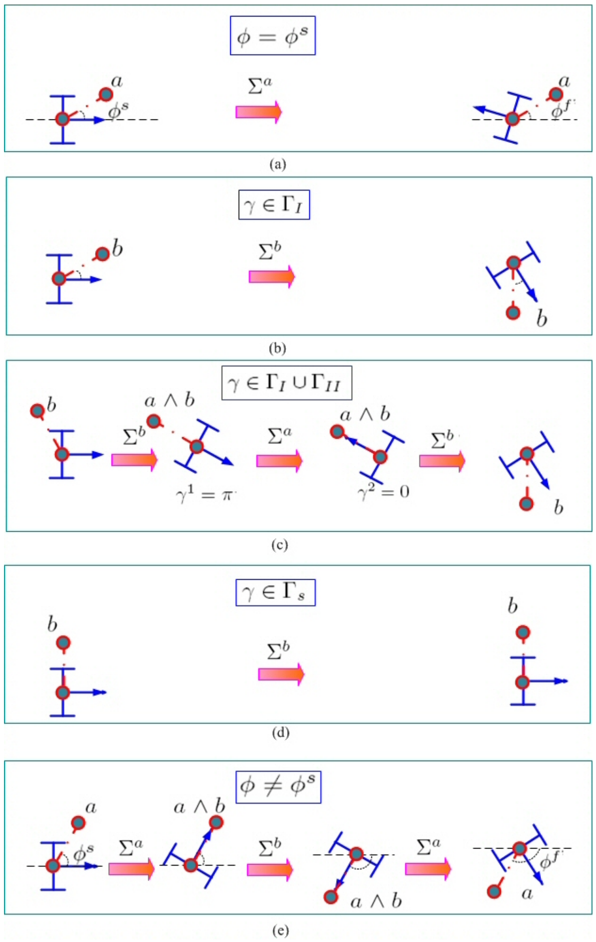

Fig. 2. The steps for deriving the reachable set of system Σ_{D^0}.

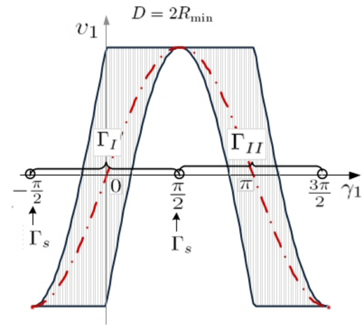

Fig. 3. Admissible control $v_1 \in U^1_{D^0}$ with respect to γ for Σ^b.

A. Reachable set in $\mathcal{M}^a$

Referring to Σ^a in (4), we have $\dot{\phi} = 0$. Hence, aircraft fly maintaining the same angular velocity (parallel Dubins paths). From the controllability of a Dubins vehicle (c.f [2]), we can steer the system between any two configurations with constant ϕ. Hence, a preliminary result is the following.

Lemma 1: The reachable set $\mathcal{R}(q^s)$ with $q^s \in \mathcal{M}^a$ contains all configuration in $\mathcal{M}^a$ with $\phi \equiv \phi^s$, see fig.2 (a).

B. Reachable set in $\mathcal{M}^b$

Let $\tilde{\Sigma}^b$ be the 4-dimensional system obtained from Σ^b projecting on the first four coordinates. Consider new coordinates for $\tilde{\Sigma}^b$ as $\tilde{q} = (x_1, y_1, \theta_1, \gamma)$ with

$$\gamma = \phi - \theta_1. \tag{8}$$

The associated vector fields are

$$g_1 = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \\ \sin\gamma \end{pmatrix};\ g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \tag{9}$$

Therefore, the kinematic model of $\tilde{\Sigma}^b$ is

$$\dot{\tilde{q}} = g_1 + g_2 v_1. \tag{10}$$

where, from (5), $v_1 \in U^1 = [\max\{-1, 2\sin\gamma - 1\}, \min\{1 + 2\sin\gamma, 1\}]$. Fig. 3 shows U^1 with respect to $\gamma \in \Gamma = [-\frac{\pi}{2}, \frac{3\pi}{2}[= \Gamma_I \cup \Gamma_{II} \cup \Gamma_s$, where

$$\Gamma_I =]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[,\ \ \Gamma_{II} =]\tfrac{\pi}{2}, \tfrac{3\pi}{2}[,\ \ \Gamma_s = \{-\tfrac{\pi}{2}, \tfrac{\pi}{2}\}. \tag{11}$$

Correspondingly, we denote $\tilde{\mathcal{M}}^b_I = \mathbb{R} \times \mathcal{S}^1 \times \Gamma_I$, $\tilde{\mathcal{M}}^b_{II} = \mathbb{R} \times \mathcal{S}^1 \times \Gamma_{II}$, and $\tilde{\mathcal{M}}^b_s = \mathbb{R} \times \mathcal{S}^1 \times \Gamma_s$.

Let $\dot{x} = f(x, u)$, where $x \in \mathcal{X} \subseteq \mathbb{R}^n$ and $u \in \mathcal{U} \subseteq \mathbb{R}^m$. Let $\mathcal{A} := \{f_u = f(., u), u \in \mathcal{U}\}$ be the set of system's vector fields.

Definition 1: The Lie algebra $\mathcal{A}_{LA}$ of vector fields $\mathcal{A}$ is called the *accessibility Lie algebra* associated to the system. The *accessibility rank condition (ARC)* holds at $x_0 \in \mathcal{X}$ if $\mathcal{A}_{LA}(x_0) = \mathbb{R}^n$.

Accessibility rank condition in [16] is also called controllability rank condition in [17], and Lie algebra rank condition in [2] and [18].

Definition 2: A system with state space $\mathcal{M}$ is *weakly reversible* if $q_1 \in \mathcal{R}(q_0) \Leftrightarrow q_0 \in \mathcal{R}(q_1)$, $\forall q_1, q_0 \in \mathcal{M}$.

Theorem 1: For a weakly reversible system, if the accessibility rank condition holds at every state $q_0 \in \mathcal{M}$ and $\mathcal{M}$ is connected, then the system is completely controllable.

The theorem follows straightforward from results in [16].

Proposition 1: For $\tilde{\Sigma}^b$, ARC holds at $\tilde{q} \in \tilde{\mathcal{M}}^b_I \cup \tilde{\mathcal{M}}^b_{II}$; but ARC fails at $\tilde{q} \in \tilde{\mathcal{M}}^b_s$.

Proof: Given $\tilde{\Sigma}^b$, for any configuration $\tilde{q}$, we take $v_\alpha, v_\beta \in U^1$. Considering two distinct control values $v_1 = v_\alpha$ and $v_1 = v_\beta$, from (9), we have the following vector fields:

$$f_1(\tilde{q}) = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ v_\alpha \\ \sin\gamma - v_\alpha \end{pmatrix}; f_2(\tilde{q}) = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ v_\beta \\ \sin\gamma - v_\beta \end{pmatrix}. \tag{12}$$

Computing the Lie brackets, we obtain:

$$f_3(\tilde{q}) = [f_1(\tilde{q}), f_2(\tilde{q})] = \begin{pmatrix} -(v_\alpha - v_\beta)\cos\theta_1 \\ (v_\alpha - v_\beta)\sin\theta_1 \\ 0 \\ -(v_\alpha - v_\beta)\cos\gamma \end{pmatrix};$$

$$f_4(\tilde{q}) = [f_1(\tilde{q}), f_3(\tilde{q})] = \begin{pmatrix} v_\alpha(v_\beta - v_\alpha)\cos\theta_1 \\ v_\alpha(v_\beta - v_\alpha)\sin\theta_1 \\ 0 \\ (v_\beta - v_\alpha)(v_\alpha\sin\gamma - 1) \end{pmatrix}.$$

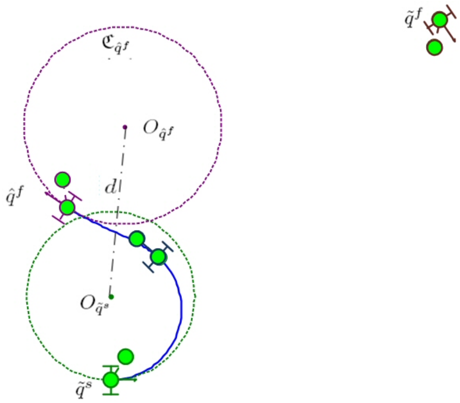

Fig. 4. The trajectory always exists from $\tilde{q}^s$ to $\hat{q}^f$ which has a same γ as $\tilde{q}^f$ for Σ^b on $\tilde{M}_I^b$.

Notice that $\det(f_1(\tilde{q}), f_2(\tilde{q}), f_3(\tilde{q}), f_4(\tilde{q})) = -(v_\beta - v_\alpha)^3$ at any q. Hence, if $v_\alpha \neq v_\beta$, $\text{rank}(f_1(\tilde{q}), f_2(\tilde{q}), f_3(\tilde{q}), f_4(\tilde{q})) = 4$.

From (11), when $\gamma \in \Gamma_I \cup \Gamma_{II}$ there always exists two different controls $v_1 = v_\alpha$ and $v_1 = v_\beta$ with $v_\alpha \neq v_\beta$, see fig.3. Hence, ARC holds at $\tilde{q} \in \tilde{\mathcal{M}}_I^b \cup \tilde{\mathcal{M}}_{II}^b$. On the contrary, when $\gamma \in \Gamma_s$ only one feasible control $v_1 = 1$ for $\gamma = \frac{\pi}{2}$ ($v_1 = -1$ for $\gamma = \frac{\pi}{2}$) exists, hence ARC fails at $\tilde{q} \in \tilde{\mathcal{M}}_s^b$. ■

To study the reachable set from points in $\mathcal{M}^b$ we first start considering $\mathcal{R}(\tilde{q}^s)$ with $\tilde{q}^s \in \tilde{\mathcal{M}}_I^b$. Similar reasonings can be applied if $\tilde{q}^s \in \tilde{\mathcal{M}}_{II}^b$.

Let $\tilde{q}^s = (x_1^s, y_1^s, \theta_1^s, \gamma^s)$ and $\tilde{q}^f = (x_1^f, y_1^f, \theta_1^f, \gamma^f)$ be initial and final configurations with $\tilde{q}^s, \tilde{q}^f \in \mathcal{M}_I^b$. We want to prove that there always exists a trajectory from $\tilde{q}^s$ to $\tilde{q}^f$ that evolves in $\mathcal{M}_I^b$, see fig.2 (b).

A preliminary result is the following.

Proposition 2: From any $\tilde{q}^s \in \mathcal{M}_I^b$, for any $\gamma^f \in \Gamma_I$ there exist x_1, y_1, θ_1 and a control that steers the system in $\mathcal{M}_I^b$ from $\tilde{q}^s$ to $\hat{q}^f = (x_1, y_1, \theta_1, \gamma^f)$, see fig.4.

Proof: Referring to fig.3, starting from $\tilde{q}^s$ and applying a control $v_1 \in U_{D^0}^1$, we are able to steer γ from γ^s to γ^f. Indeed, given $\gamma^s \in \Gamma_I$ with $\gamma^s \geq 0$ ($\gamma^s < 0$) it is possible to apply $v_1 = 1$ ($v_1 = -1$) until $\gamma = 0$ then $v_1 = 1 + 2\sin\gamma$ ($v_1 = 2\sin\gamma - 1$) until $\gamma = \gamma^f$ if $\gamma^f \leq 0$ ($\gamma^f > 0$). Otherwise, from $\gamma^s \geq 0$ ($\gamma^s < 0$), apply $v_1 = 1$ ($v_1 = -1$) until $\gamma = 0$, and then switch to $v_1 = 2\sin\gamma - 1$ ($v_1 = 1 + 2\sin\gamma$) until $\gamma = \gamma^f$ if $\gamma^f \geq 0$ ($\gamma^f < 0$). The point reached with such control law is a point $\hat{q}^f = (x_1, y_1, \theta_1, \gamma^f)$ for some x_1, y_1, θ_1, see fig.4. ■

Notice that we chose the control law to let γ be zero at least one time when steering from γ^s to γ^f. Also notice that, when $\gamma = 0$, Dubins aircraft can move along a straight line while when $v_1 = \sin\gamma$, the trajectory is a circle $\mathfrak{C}$ of radius $|v_1^{-1}|$ along which γ is constant. Let $\mathfrak{C}_{\tilde{q}}$ be the trajectory circle through $\tilde{q} = (x_1, y_1, \theta_1, \gamma)$ with radius $\left|\frac{1}{\sin\gamma}\right|$ and let $O_{\tilde{q}}$ be its center.

Theorem 2: There always exists a trajectory from $\tilde{q}^s$ to $\tilde{q}^f$ that evolves in $\mathcal{M}_I^b$.

Proof: Consider $\mathfrak{C}_{\tilde{q}^s}$ with center $O_{\tilde{q}^s}$ and $\mathfrak{C}_{\tilde{q}^f}$ with center $O_{\tilde{q}^f}$. Let δ be the distance between $O_{\tilde{q}^s}$ and $O_{\tilde{q}^f}$. Let $\hat{q}^f = (x_1, y_1, \theta_1, \gamma^f)$ for some x_1, y_1, θ_1 be the final point reached with the control law used in the proof of Proposition 2. For the chosen control law there always exists a point along the trajectory from $\tilde{q}^s$ to $\hat{q}^f$ with $\gamma = 0$. Hence, the distance d between $O_{\tilde{q}^s}$ and $O_{\hat{q}^f}$ can be continuously increased by keeping $v_1 = 0$ at $\gamma = 0$ as long as it is needed. Two cases are now possible:

1) If $\delta \geq d$ it is possible to generate a control law such that the final point $\hat{q}^f$ of the new trajectory T is such that $\delta = d$. There exist a rotation centered in $O_{\tilde{q}^s}$ that moves $O_{\hat{q}^f}$ in $O_{\tilde{q}^f}$ and $\mathfrak{C}_{\hat{q}^f}$ on $\mathfrak{C}_{\tilde{q}^f}$, see fig.5 (a). The trajectory from $\tilde{q}^s$ to $\tilde{q}^f$ consists in a circle arc on $O_{\tilde{q}^s}$ followed by T and a circle arc on $O_{\hat{q}^f}$, shown in fig.5 (b).
2) If $\delta < d$ it is possible to reach a point $\hat{q}^f$ such that the distance between $O_{\hat{q}^f}$ and $O_{\tilde{q}^f}$ is arbitrary large. Hence, using the similar reasoning and the result of above case $\delta \geq d$, a trajectory from $\hat{q}^f$ to $\tilde{q}^f$ can be obtained.

■

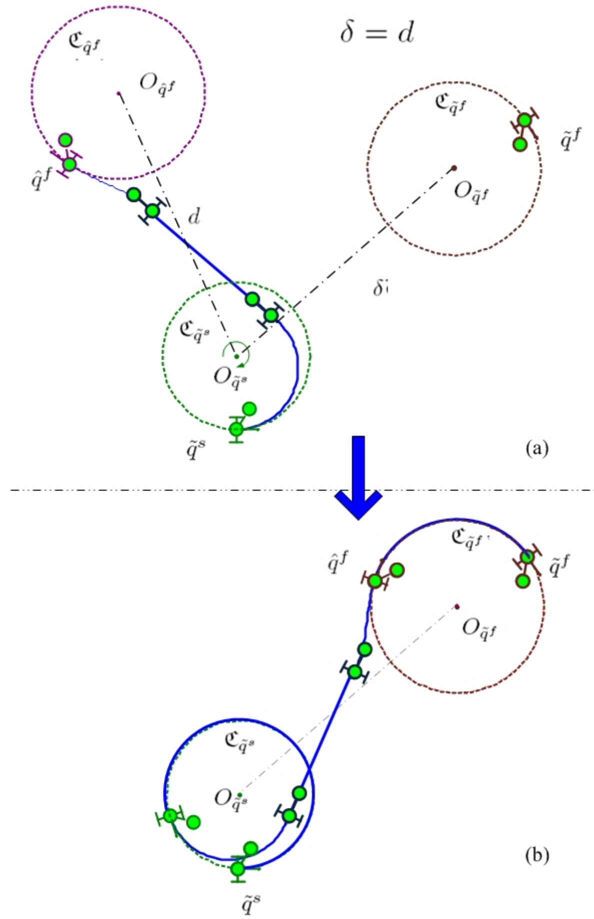

Fig. 5. (a) The trajectory is extended for $\delta = d$, (b) The trajectory exists from $\tilde{q}^s$ to $\tilde{q}^f$ by beginning with an arc of $\mathfrak{C}_{\tilde{q}^s}$ and ending with an arc of $\mathfrak{C}_{\tilde{q}^f}$.

Theorem 2 implies that Σ^b is a weakly reversible system for $\tilde{q} \in \tilde{\mathcal{M}}_I^b$. The same reasoning can be applied to $q \in \tilde{\mathcal{M}}_{II}^b$.

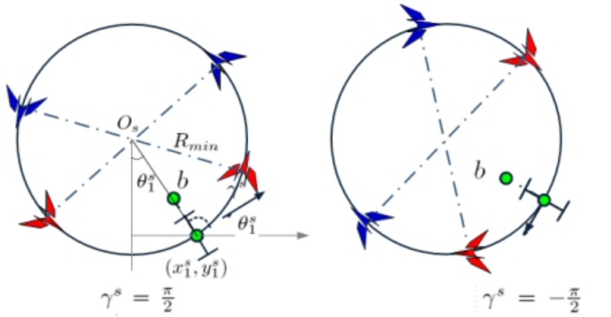

Fig. 6. The reachable configurations are on the limited circles with $\gamma \in \Gamma_s$ of $\tilde{\Sigma}^b$.

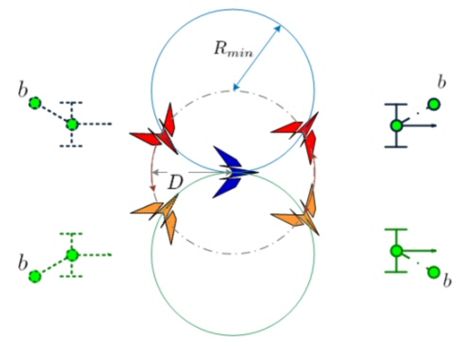

Fig. 7. An example ($D = R_{\min}$) for the feasible configurations of system Σ_{D-}. The feasible positions of aircraft 2 are on the thick red curves for angular relation b.

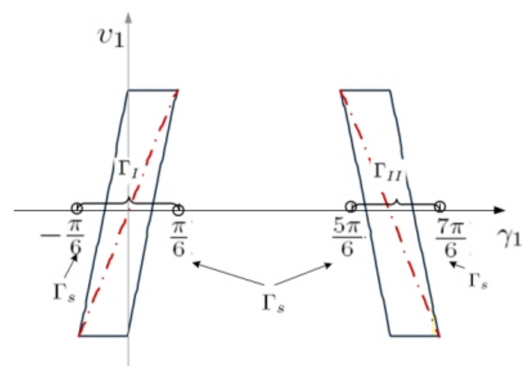

Fig. 8. An example ($D = R_{\min}$) for the admissible controls $v_1 \in U^1_{D-}$ with respect to γ_1 for system Σ^b.

C. Reachable set for Σ_{D^0}

Going back to the 5-dimensional space, let $\mathcal{M}^b_I$, $\mathcal{M}^b_{II}$ and $\mathcal{M}^b_s$ be the subsets of $\mathcal{M}^b_{D_0}$ whose projection on the 4-dimensional space are $\tilde{\mathcal{M}}^b_I$, $\tilde{\mathcal{M}}^b_{II}$ and $\tilde{\mathcal{M}}^b_s$ respectively.

Theorem 3: The reachable set $\mathcal{R}(q^s)$ for system Σ_{D^0} from configuration $q^s \in \mathcal{M}^b_I \cup \mathcal{M}^b_{II}$ contains $\mathcal{M}^b_I \cup \mathcal{M}^b_{II}$.

Proof: From Theorem 1, Proposition 1, Theorem 2 and the fact that $\tilde{\mathcal{M}}^b_I$ is connected we obtain that the reachable set $\mathcal{R}(q^s)$ for system Σ^b with $q^s \in \mathcal{M}^b_I$ ($q^s \in \mathcal{M}^b_{II}$) contains $\mathcal{M}^b_I$ ($\mathcal{M}^b_{II}$). Indeed the fifth component θ_2 depends on θ_1 and angular relation b.

Without loss of generality, we assume $q^s \in \mathcal{M}^b_{II}$ and $q^f \in \mathcal{M}^b_I$ (the vice versa can be solved equivalently). From q^s, point $q^1 = (x^1_1, y^1_1, \theta^1_1, \gamma^1, 2\gamma^1 + \theta^1_1) \in \mathcal{M}^b_{II}$ with $\gamma^1 = \pi$ is reachable, as shown in fig.2 (c). From q_1, the system evolves as Σ^a, by maintaining $\phi(t) \equiv \phi^1 = \gamma^1 + \theta^1$, so that the trajectory can reach a configuration $q^2 = (x^2_1, y^2_1, \theta^2_1, \gamma^2, \theta^2_1)$ with $\gamma^2 = 0$. Finally, letting the system evolve as Σ^b, q^f can be reached. In the same way, we can find a feasible trajectory from $\forall q^s \in \mathcal{M}^b_I$ to $\forall q^f \in \mathcal{M}^b_{II}$. ■

Lemma 2: The reachable set $\mathcal{R}(q^s)$ for $q^s \in \mathcal{M}^b_s$ lays on a circle with radius $R_{\min}$ and centered at $O_s : (x^s_1 - R_{\min} \sin\theta^s_1 \sin\gamma^s, y^s_1 + R_{\min} \cos\theta^s_1 \sin\gamma^s)$ along which $\gamma = \gamma^s$.

Proof: If $\gamma^s = \frac{\pi}{2}$ ($\gamma^s = -\frac{\pi}{2}$), $v_1 \equiv 1$ ($v_1 \equiv -1$) is the unique admissible control value and reachable configurations lay on a circle through q^s with radius $R_{\min}$, see fig.6. ■

Finally we are able to extend Lemma 1 proving that

Lemma 3: The reachable set for system Σ_{D^0} from $q^s \in \mathcal{M}^a$ contains $\mathcal{M}^a$.

Proof: For $q^s = (x^s_1, y^s_1, \theta^s_1, \gamma^s, \theta^s_1) \in \mathcal{M}^a$, Lemma 1 implies that any configuration with $\phi = \phi^s$ can be reached in $\mathcal{M}^a$. Referring to fig.2 (e), if the final configuration in $\mathcal{M}^a$ is such that $\phi \neq \phi^s$ we proceed as follows: 1) from q^s, achieve a configuration $q^1 = (x^1_1, y^1_1, \theta^1_1, \gamma^1)$ with $\theta^1_1 = \phi^1 = \phi^s$, notice that $q^1 \in \mathcal{M}^b$ with $\gamma^1 = 0$. 2) From q^1 it is possible to reach q^2 with $\theta^2_1 = \phi^2 = \phi^f$, evolving with Σ^b from Theorem 3. 3) From q^2, q^f can be reached evolving according to Σ^a (Lemma 1). ■

Concluding, for system Σ_{D^0} we have the following controllability property:

Theorem 4: Σ_{D^0} is controllable on the configuration space $\mathcal{M}_{D^0} = \mathcal{M}^a \cup \mathcal{M}^b_I \cup \mathcal{M}^b_{II}$. But, if $q \in \mathcal{M}^b_s$, the reachable configurations are on a limit circle.

IV. Controllability for Σ_{D-}

Let U^1_{D-} denote the admissible control set U^1 when $D < 2R_{\min}$. According to (6) and (8), if v_1 is admissible, it has to satisfy the following inequalities, $\frac{4\sin\gamma}{D} - 1 \leq 1$ and $1 + \frac{4\sin\gamma}{D} \geq -1$, hence $|\sin\gamma| \leq \frac{D}{2}$.

If $D < 2R_{\min}$, $\gamma_1 \in \Gamma_I \cup \Gamma_{II} \cup \Gamma_s$, where

$$\begin{aligned} \Gamma^-_I &=]-\arcsin(\tfrac{D}{2}), \arcsin(\tfrac{D}{2})[, \\ \Gamma^-_{II} &=]-\arcsin(\tfrac{D}{2}) + \pi, \arcsin(\tfrac{D}{2}) + \pi[, \\ \Gamma^-_s &= \{\gamma_1 | |\sin\gamma_1| = \tfrac{D}{2}\}. \end{aligned} \tag{13}$$

For example, for $D = R_{\min}$ feasible configurations are represented in fig. 7 and admissible controls ($\Gamma^-_I =]-\frac{\pi}{6}, \frac{\pi}{6}[$, $\Gamma^-_{II} =]\frac{5\pi}{6}, \frac{7\pi}{6}[$, and $\Gamma^-_s = \{-\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}\}$) are represented in fig. 8.

We denote with $\mathcal{M}^b_{D-} = \mathbb{R}^2 \times \mathcal{S}^1 \times \{\Gamma^-_I \cup \Gamma^-_{II}\} \times \mathcal{S}^b$ and $\mathcal{M}^{b^s}_{D-} = \mathbb{R}^2 \times \mathcal{S}^1 \times \Gamma^-_s \times \mathcal{S}^b$.

Proposition 3: (Configuration Space of System Σ_{D-}) When $D < 2R_{\min}$, let $(x_1, y_1, \theta_1, \phi, \theta_2)$ be a configuration of Σ_{D-}, then its configuration space is $\mathcal{M}_{D-} = \mathcal{M}^a \cup \mathcal{M}^b_{D-} \cup \mathcal{M}^{b^s}_{D-}$.

Theorem 5: (Reachable set of system Σ_{D-}) Σ_{D-} is controllable on the configuration space $\mathcal{M}_{D-} = \mathcal{M}^a \cup \mathcal{M}^b_{D-}$. But the reachable configurations are on a limit circle, if $q \in M^{b_s}_{D-}$.

Proof: The proof can be done applying the reasoning used for deriving the reachable sets of system Σ_{D^0}. If the system evolves according to Σ^a, Σ_{D-} has the same evolution

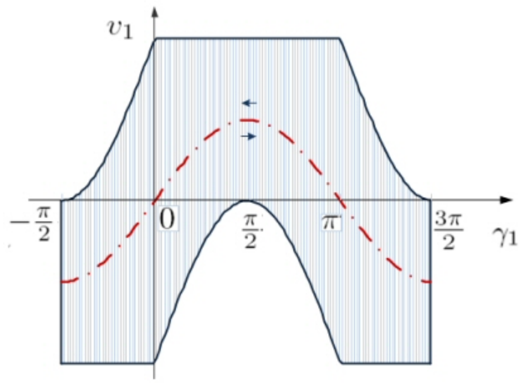

Fig. 9. An example ($D = 4R_{\min}$) for the admissible controls $v_1 \in U^1_{D+}$ with respect to γ_1 for system Σ^b.

equations as Σ_{D^0}. The difference between the two systems is in Σ^b because of the different admissible controls. ARC property fails on $q^s \in \mathcal{M}^{b_s}_{D-}$ with $|\sin\gamma^s| = \frac{D}{2}$, hence the thesis. ■

V. Controllability for Σ_{D+}

When $D > 2R_{\min}$, we denote with $\mathcal{M}^b_{D+} = \mathbb{R}^2 \times \mathcal{S}^1 \times \mathcal{S}^1 \times \mathcal{S}^b$.

Proposition 4: (Configuration Space of System Σ_{D+}) When $D > 2R_{\min}$, let $(x_1, y_1, \theta_1, \phi, \theta_2)$ be a configuration of Σ_{D+}, then its configuration space is $\mathcal{M}_{D+} = \mathcal{M}^a \cup \mathcal{M}^b_{D+}$.

Theorem 6: (Reachable set of system Σ_{D+}) Σ_{D+} is completely controllable on the configuration space $\mathcal{M}_{D+}$.

Proof: The proof can be done applying the reasoning used for deriving the reachable sets of system Σ_{D-}. Let U^1_{D+} denote admissible control set U^1 when $D > 2R_{\min}$ fig.9 illustrates an example of the admissible controls with respect to γ, where $D = 4R_{\min}$. For Σ_{D+}, at any $q \in \mathcal{M}_{D+}$, there exist two feasible values for v_1. From Proposition 1, ARC holds at any configuration for $\mathcal{M}^b_{D+}$. Therefore Σ_{D+} is controllable. ■

VI. Controllability for Aircraft Formations

Based on results obtained from the case of two aircraft, we now study controllability properties of the multi-aircraft formations. In particular we will consider formations in which n aircraft maintain a constant distance from a single reference one (star formation), or between consecutive pairs of aircraft, both open (chain formation) and closed (ring formation). We do not consider possible collisions between aircraft (as if e.g. they were actually flying on different altitude layers).

From the discussion above, we already know that, already for $n = 2$, complete controllability does not hold if the distance between the aircraft is not larger than twice the minimum radius of rotation. Hence, in what follows we will assume that for every pair in the formation, it holds $D_i > 2R_{\min}$.

A. Controllability for star formations

Let Σ^n_s be the system of n aircraft V_i, $i = 1, \cdots, n$, where the distances to be maintained are the distances D_i between

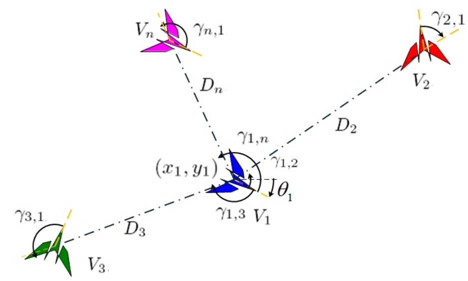

Fig. 10. The star formation for n aircraft.

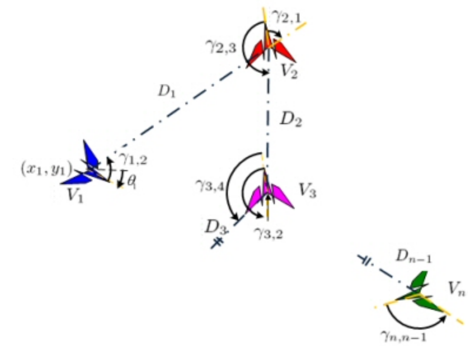

Fig. 11. The chain formation for n aircraft.

V_1 and V_i, $i = 2, \cdots, n$. Let $\gamma_{1,i}$ ($\gamma_{i,1}$), $i = 2, \cdots, n$ denote the angle from the heading direction of V_1 (V_i) to the distance direction from V_1 to V_i, see fig.10. A configuration $\bar{q}$ of Σ^n_s is $\bar{q} = (x_1, y_1, \theta_1, \gamma_{1,2}, \gamma_{2,1}, \cdots, \gamma_{1,n}, \gamma_{n,1})$. Let S^a be a subset of $\mathcal{S}^1$ with angular relation $a : \gamma_{1,i} = \gamma_{i,1}$ and S^b be a subset of $\mathcal{S}^1$ with angular relation $b : \gamma_{1,i} = -\gamma_{i,1}$. The configuration space of Σ^n_s is therefore $\mathcal{M}_s = \mathbb{R}^2 \times \mathcal{S}^1 \times \left(\mathcal{S}^1 \times (S^a \cup S^b)\right)^{n-1}$.

A simple result for controllability is obtained in the assumption that distances are sufficiently large, namely $D_i > 4R_{\min}$, $i = 2, \cdots, n$. Indeed in this case the set of admissible v_1, described by $-1 + \max\{0, 4\sin\gamma_{1,2}/D_2, \cdots, 4\sin\gamma_{1,n}/D_n\} \le v_1 \le 1 + \min\{4\sin\gamma_{1,2}/D_2, \cdots, 4\sin\gamma_{1,n}/D_n, 0\}$ contains an open subset in $\mathbb{R}^1$ for any $\gamma_{1,i}$, hence two distict controls can always be applied, and ARC follows froma direct extension of the Lie algebra calculations reported above. Weak reversibility can be shown constructively by the motion planning algorithm proposed in [19], hence the claim of complete controllability in this case.

If $2R_{\min} < D_i \le 4R_{\min}$ for some i, there exist configurations of the formation for which no admissible controls exist that could keep the formation. A detailed study of the reachable sets in this case is rather complex, and is the subject of further studies.

B. Controllability for chain formations

Let Σ^n_c be the system of n aircraft V_i, $i = 1, \cdots, n$ that maintain constant distances D_i, $i = \mathcal{I} = \{1, \cdots, n-1\}$ between V_i and V_{i+1}, with $D_i > 2R_{min}$ $\forall i$. Let $\gamma_{i,i+1}$ ($\gamma_{i+1,i}$), $i = 1, \cdots, n-1$ denote the angle from the heading direction of V_i (V_{i+1}) to the line from V_i to V_{i+1}, see fig.11.

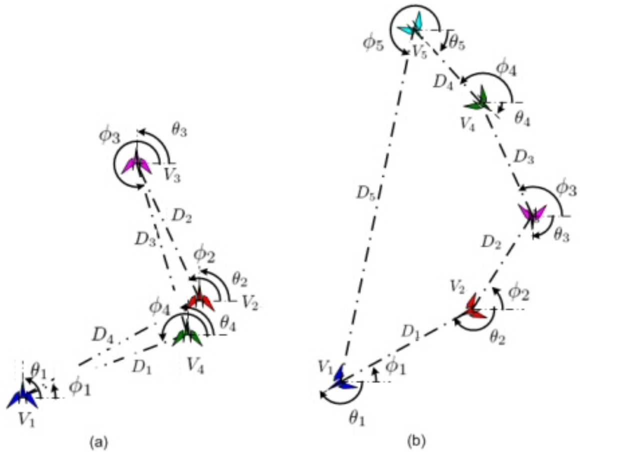

Fig. 12. For aircraft ring formation, the system configurations with angular relation a maintain the formations for any polygons: (a) concave quadrilateral; (b) convex pentagon.

Let $\mathcal{I}^b$ be the set of indices $i \in \mathcal{I}$ such that the angles of V_i and V_{i+1} satisfy condition b.

A configuration $\bar{q}$ of Σ_c^n is $\bar{q} = (x_1, y_1, \theta_1, \gamma_{1,2}, \gamma_{2,1}, \cdots, \gamma_{n-1,n}, \gamma_{n,n-1})$. Let S^a be a subset of $\mathcal{S}^1$ with angular relation a : $\gamma_{i,i+1} = \gamma_{i+1,i}$ and S^b be a subset of $\mathcal{S}^1$ with angular relation b : $\gamma_{i,i+1} = -\gamma_{i+1,i}$. The configuration space of Σ_c^n is $\bar{\mathcal{M}}_c = \mathbb{R}^2 \times \mathcal{S}^1 \times \left(\mathcal{S}^1 \times (S^a \cup S^b)\right)^{n-1}$.

Consider first the case $\mathcal{I} = \mathcal{I}^b$, and let v_i be the angular velocity of V_i. By applying (5) recursively along any sub-chain, we have $v_k = \sum_{i=k}^{j-1} (-1)^{i-k} \frac{4}{D_i} \sin \gamma_{i,i+1} + (-1)^{j-k} v_j$, $\forall j \in \{1, \ldots, n-1\}$, $\forall k \in \{j+1, \ldots, n\}$. A necessary and sufficient condition to ensure the existence of an open control set hence is

$$\left| \sum_{i=k}^{j-1} (-1)^{i-k} \frac{4}{D_i} \sin \gamma_{i,i+1} \right| < 2, \text{ for any } j, k : 1 \le k < j \le n.$$

Again, a simple result for controllability is obtained in the assumption that distances are sufficiently large. Under the hypothesis that $\sum_{i=k}^{n-1} \frac{1}{D_i} < \frac{1}{2}$, the set of admissible controls for every chain formation with $\mathcal{I} = \mathcal{I}^b$ contains an open set, hence the ARC holds. When $\mathcal{I}^b \subset \mathcal{I}$, i.e. there exist aircraft pairs satisfying condition a, the condition above is sufficient *a fortiori* (indeed, condition a imposes a simple constraint on the velocities, i.e. $v_i = v_{i+1}$, and may only reduce the length of the summation to compute v_k reported above). Weak reversibility can be shown constructively by the motion planning algorithm described in [19].

C. Controllability for ring formations

Let Σ_r^n be the system of n aircraft $V_i,\ i = 1, \cdots, n$, that maintain constant distance $D_i,\ i = 1, \cdots, n$ between V_i and V_{i+1}, where V_{n+1} denotes aircraft V_1. Let $\phi_i,\ i = 1, \cdots, n$ be the angle of vector from the position of V_i to the position of V_{i+1} with respect to the x-axis, see fig. 12. A configuration $\bar{q}$ of Σ_r^n is $\bar{q} = (x_1, y_1, \theta_1, \phi_1, \theta_2, \phi_2 \cdots, \theta_n, \phi_n)$. If for all pairs of aircraft, angular relation a : $\theta_i = \theta_{i+1} \wedge \theta_1 = \theta_n$ holds, then all θ_i are equal, see fig.12. The reachable configurations can be any x_1, y_1 and any equivalent $\theta_i, i = 1, \cdots, n$ with constant $\phi_i = \phi_i^s, i = 1, \cdots, n$ from $\bar{q}^s = (x_1^s, y_1^s, \theta_1^s, \phi_1^s, \theta_1^s, \phi_2^s \cdots, \theta_1^s, \phi_n^s)$, i.e. all aircraft travel following a parallel Dubins path and maintain the form of the initial polygon which has been determined by $\bar{q}^s$.

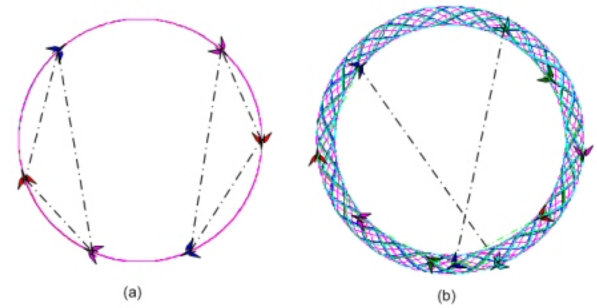

Fig. 13. All reachable configurations are located on the limited trajectories shown in (a) for $n = 3$ and (b) for $n = 5$.

If all pairs of aircraft have angular relation b, then the following equations must be satisfied:

$$\begin{cases} \theta_{i+1} = 2\phi_i - \theta_i, \text{ for } i = 1, \ldots, n-1 \text{ and} \\ \theta_1 = 2\phi_n - \theta_n \end{cases} \tag{14}$$

The behaviors of systems Σ_r^n are totally different for even or odd n.

1) Odd number of aircraft:
If n is odd and each pair of aircraft has angular relation b, from (14), we have: $\theta_1 = \sum_{i=1}^{(n+1)/2} \phi_{2i-1} - \sum_{i=1}^{(n-1)/2} \phi_{2i}$; $\theta_2 = \sum_{i=1}^{(n-1)/2} \phi_{2i} + \phi_1 - \sum_{i=1}^{(n-1)/2} \phi_{2i+1}$; $\cdots$; $\theta_n = \phi_n + \sum_{i=1}^{(n-1)/2} \phi_{2i} - \sum_{i=1}^{(n-1)/2} \phi_{2i-1}$. It implies that all feasible $\theta_i, i = 1, \cdots, n$ can be uniquely determined by $\phi_i, i = 1, \cdots, n$ and all admissible angular controls v_i for aircraft V_i can be uniquely determined by $\dot{\phi}_i, i = 1, \cdots, n$. $\dot{\phi}_i = \frac{4 \sin(\phi_i - \theta_i)}{D_i}$ implies that for a given configuration $\bar{q}$, only one admissible control value v_i exists.

Fig.13 shows all reachable configurations with $n = 3$ and $n = 5$ respectively. If $n = 3$, three aircraft forms a triangle. We can prove that $v_1 = v_2 = v_3 = -2\frac{\sin \beta_3}{D_1} = -2\frac{\sin \beta_1}{D_2} = -2\frac{\sin \beta_2}{D_3}$, where β_i denotes the corresponding angle of V_i as vertex of the triangle. The circumscribed circle has radius $r_t = \frac{D_1}{2 \sin \beta_3} = \frac{D_2}{2 \sin \beta_1} = \frac{D_3}{2 \sin \beta_2}$. Thus the feasible directions for three aircraft are tangent to the circumscribed circle of the triangle. A feasible trajectory is this circle along which aircraft travel anticlockwise or clockwise with angular velocity $2\frac{|\sin \beta_3|}{D_1}$, as shown in fig.13 (a).

If $n > 3$, all reachable configurations are located on a limited trajectory (see e.g. fig.13 (b)). Note that unlike $n = 3$, the form of a polygon changes along its only feasible trajectory. If a ring formation is not on one line, there is no shared case between angular relations a and b.

2) Even number of aircraft:

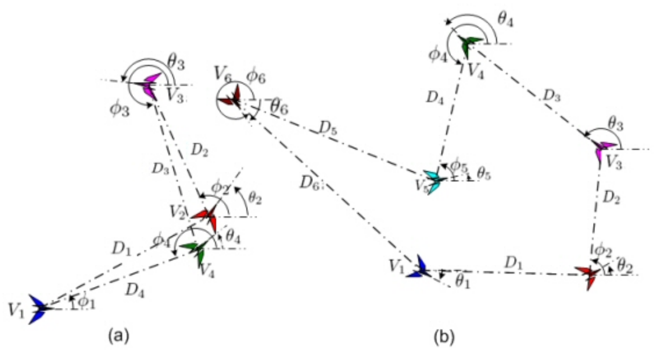

Fig. 14. The ring formations for (a) $n = 4$ and (b) $n = 6$.

If n is even and each pair of aircraft has angular relation b, from (14), we have:

$$\sum_{i=1}^{n/2} \phi_{2i-1} = \sum_{i=1}^{n/2} \phi_{2i}. \qquad (15)$$

Thus any polygon of even n with angular relation b must satisfy the condition (15). Moreover admissible control $\mathcal{U}^1 \neq \emptyset$ on feasible configurations. We can get $\mathcal{U}^1 = [-1 + \max\{0, \frac{4\sin(\phi_1-\theta_1)}{D_1}, \frac{4\sin(\phi_1-\theta_1)}{D_1} - \frac{4\sin(\phi_2-\theta_2)}{D_2}, \cdots, \sum_{i=1}^{n-1}(-1)^{i-1}\frac{4\sin(\phi_i-\theta_i)}{D_i}, \frac{4\sin(\phi_n-\theta_n)}{D_n}\}, 1 + \min\{0, \frac{4\sin(\phi_1-\theta_1)}{D_1}, \frac{4\sin(\phi_1-\theta_1)}{D_1} - \frac{4\sin(\phi_2-\theta_2)}{D_2}, \cdots, \sum_{i=1}^{n-1}(-1)^{i-1}\frac{4\sin(\phi_i-\theta_i)}{D_i}, \frac{4\sin(\phi_n-\theta_n)}{D_n}\}$. Fig.14 illustrates two examples of $n = 4$ and $n = 6$ ring formations with angular relation b. Moreover the reachable configurations for ring formations with even n are contained in a limited area and satisfy conditions (15) and $\mathcal{U}^1 \neq \emptyset$.

A ring formations can have mixed angular relation a and b. From (14), the behavior of such systems depends on the number of the pairs with angular relation b. If the number of pairs with b is n_b, the system Σ_n runs as the ring formation Σ_{n_b} for the n_b pairs while the other pairs move in parallel (same angular velocity).

VII. Conclusions

In this paper the configuration spaces and controllability results for the systems of n identical Dubins aircraft formations maintaining constant distances have been provided. First a system of a pair of airplanes maintaining a constant distance D has been studied. The proposed study proves that the system is controllable when $D > 2R_{\min}$, while limit circles for particular configurations have been proved to exist when $D \leq 2R_{\min}$.

Controllability results for three basic formations of n aircraft (star formations, chain formations and ring formations) are provided. The systems consisting is star formations are completely controllable when all given distance are $D \geq 4R_{\min}$. The chain formation systems are completely controllable if given distances satisfy $\sum_{i=1}^{n-1} \frac{1}{D_i} \leq \frac{1}{2}$. For the ring formations at most two cases exist. For any polygon formed by aircraft in ring formations, all vehicles can travel parallel to each other as a single Dubins vehicle. In addition, if some aircraft in the formation verify angular relation b, the reachable configurations are contained in a uniquely determined trajectory when n is an odd number, while the reachable configurations are contained in a limited area when n is an even number.

Based on the results of this paper on the controllability of aircraft formations, we provided the results of the motion planning of formation of aircraft in [19]. The results of controllability of aircraft formations also provide important conditions for the proof of the existence of optimal formation trajectories.

VIII. Acknowledgments

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